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MATHEMATICAL REVIEWS

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References to reviews in Mathematical Reviews before volume 20 (1959) are by volume and page number, as MR **19**, 532: from volume 20 on, by volume and review number, as MR **20** #4387. Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parentheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.

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Mathematical Reviews

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GENERAL

See also 4820.

4614:

Ringleb, Friedrich. ★Mathematische Formelsammlung. 7te, erweiterte Aufl. Sammlung Göschen, Bd. 51/51a. Walter de Gruyter & Co., Berlin, 1960. 320 pp. DM 5.80.

[Sixth edition (1956) is listed in MR **18**, 181.] Contents: Arithmetic and combinatorial analysis; algebra; number theory; elementary series; plane geometry; 3-dimensional geometry; plane trigonometry; spherical trigonometry; geography and astronomy; plane analytic geometry; 3-dimensional analytic geometry and vectors; differential calculus; integral calculus; function theory and conformal mapping; differential geometry; differential equations.

4615:

Lasker, Edward. ★Go and Go-Moku: The Oriental board games. 2nd revised ed. Dover Publications, Inc., New York, 1960. xix + 215 pp. $1.45.

The original edition, published in 1934, was a well-known introduction to Go for Occidental novices. The author's revision is intended to incorporate as much as possible of contemporary masters' doctrine.

4616:

★Lucrările celui de al IV-lea Congres al Matematicienilor Romîni [Proceedings of the 4th Congress of Romanian Mathematicians]. Editura Academiei Republicii Populare Romîne, Bucharest, 1960. 227 pp. Lei 8.55.

Abstracts of papers presented at the congress in Bucharest, 1956. Introductory addresses are in Romanian; the abstracts themselves mostly in French, a few in German, Russian or English.

HISTORY AND BIOGRAPHY

4617:

Bruins, E. M. The algebra of antiquity and the middle ages. Euclides **34** (1958/59), 131–159. (Dutch)

4618:

Mohan, Brij. Progressions in ancient Hindu mathematics. J. Sci. Res. Banaras Hindu Univ. **9**, no. 1, 19–28 (1958/59).

The author considers the Hindu knowledge of progressions as shown by writers from Āryabhata (A.D. 499) to Bhāskara (c. A.D. 1150). (The reviewer did not have available the Paramsiddhanta, quoted by the author.) Knowledge of progressions occurs much earlier in other lands; in Babylonia and Egypt in the second millenium B.C., in Greece by Euclid [*Elements*, Book IX], and in China by the beginning of our era.

The methods discussed in this paper have long been known in the West through well-known translations such as those by Taylor in 1816, Colebrooke in 1817, and Clark in 1930, some with explanatory examples and equivalent formulas in modern notation. *E. B. Allen* (Troy, N.Y.)

4619:

Bockstaele, P. Notes on the first arithmetics printed in Dutch and English. Isis **51** (1960), 315–321.

On the earliest known arithmetic in English [*An introduction for to lerne to recken with the pen, or with the counters*, John Herford, St. Albans, 1537] and the works of which it was a compilation.

4620:

Bourbaki, Nicolas. ★Éléments d'histoire des mathématiques. Histoire de la Pensée, IV. Hermann, Paris, 1960. 277 pp. 18 NF.

This volume collects the "Notes historiques" contained in the published parts of the authors' *Éléments de Mathématiques*. Hence it does not pretend to be a well-balanced history of mathematics; topics, such as Differential geometry, Algebraic geometry, Calculus of variations, Number theory, Analytic functions, Differential equations, which have not yet been discussed extensively in the *Éléments*, are almost entirely omitted. About 90 pages deal with the Foundations of mathematics and with the Calculus. Algebra, Elementary geometry, Topology and Theory of integration together with some more specialized subjects take the other 170 pages.

In their presentation the authors have been guided by their view of modern mathematics. It is this retrospective view of the development of mathematics (outweighing the usual genetic description) which is probably the most important and unusual feature of this torso. It also accounts for the heavy emphasis given to the mathematics of the 19th and 20th century.

In summary, this book is the first part of a history of the mathematical ideas which have contributed to the modern picture of mathematics as seen by the leading group of French mathematicians.

It is very desirable that this work, which contains a bibliography of almost 400 numbers, be continued. An explanation of symbols and, in particular, a general index should prove very helpful. *C. J. Scriba* (Toronto)

4621:
Problems and perspectives of mathematics in Czechoslovakia. Časopis Pěst. Mat. **85** (1960), 129–132. (Czech)

4622:
Frigyes Riesz: necrologue. Uspehi Mat. Nauk **12** (1957), no. 4 (76), 155–166. (1 plate) (Russian)
Russian translation of the Hungarian original in Mat. Lapok **7** (1956), 1–9 [MR **20** #5112].

4623:
Léopold Fejér, 1880–1959. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **2** (1959), 3–4.

4624:
The late Academician Eduard Čech. Časopis Pěst. Mat. **85** (1960), 217. (Czech)

4625:
Kolibiar, M.; Švec, M. Academician Jur Hronec in memoriam. Časopis Pěst Mat. **85** (1960), 218–225. (1 plate) (Slovak)

4626:
Bílek, Jan. Academician Bohumil Bydžovský at eighty. Časopis Pěst. Mat. **85** (1960), 226–227. (Czech)

4627:
Škrášek, Josef. The life and work of Matyáš Lerch. Časopis Pěst. Mat. **85** (1960), 228–240. (Czech)

4628:
Pavlíček, Jan B. Centenary of the death of János Bolyai. Časopis Pěst. Mat. **85** (1960), 241–255. (Czech)

LOGIC AND FOUNDATIONS

4629:
Geymonat, Ludovico. Matematica, metamatematica e filosofia. Rend. Mat. e Appl. (5) **19** (1960), 124–129.

4630:
Newman, M. H. A. What is mathematics? New answers to an old question. Math. Gaz. **43** (1959), 161–171. (1 plate)
Presidential address to the Mathematical Association, April 1959, emphasizing the axiomatic viewpoint.

4631:
Hooley, J. Sentence logic as an introduction to axiomatic systems. Math. Gaz. **44** (1960), 23–34.
Discussion of propositional calculus, regarded as an axiomatic system, with demonstration of consistency, completeness and decidability.

4632:
Mihăilescu, Eugen. Les formes normales dans le calcul bivalent des propositions. An. Univ. "C. I. Parhon" Bucureşti. Ser. Acta Logica **2** (1959), no. 1, 201-227. (Russian and English summaries)
This study concerns the classical propositional calculus based on implication, equivalence, non-equivalence and conjunction. A formal connective D ("logical difference") is introduced by means of the axiom $EDpqRKpqp$, where E, R and K are equivalence, non-equivalence and conjunction. It is proved that any formula in which D is the only connective is equivalent to a "normal" formula $R\cdots RP$, where P is a formula with K as its sole connective. The proof includes explicit instructions for exhibiting this normal formula.
G. F. Rose (Pacific Palisades, Calif.)

4633:
Hintikka, K. Jaakko J. Notes on the quantification theory. Soc. Sci. Fenn. Comment. Phys.-Math. **17**, no. 12, 13 pp. (1955).
The compactness theorem of predicate calculus PC states that an arbitrary set (of formulae of PC) has a model if each finite subset has a model. The author observes that this result can be obtained quite simply on the basis of his analysis of single formulae in PC [Acta Philos. Fenn. **8** (1955), 7–55; MR **16**, 1079]. Actually the same applies to any of the other cut-free formalizations of PC such as those of Schütte [Arch. Math. Logik Grundlagenforsch. **2** (1956), 55–67; MR **19**, 3], Beth's semantic tableaux [Meded. Kon. Nederl. Akad. Wetensch. Afd. Letterkunde (N.R.) **18** (1955), no. 13; MR **19**, 625] and of course Gentzen's original cut-free formalization of PC. It may be remarked that, for prenex formulae, the compactness theorem is obtained straightforwardly from Herbrand's analysis as given, e.g., in Hilbert and Bernays' *Grundlagen der Mathematik*, Bd. 2 [Springer, Berlin, 1939] if one remarks that for an arbitrary infinite initial ordinal α there is a set of disparate functions [loc. cit., p. 168] φ_β with prescribed finite number n_β of arguments for each $\beta<\alpha$. It is to be noted that, while these proofs bring out interesting additional facts about PC, compactness itself is more simply proved, as in Rasiowa and Sikorski [Fund. Math. **37** (1950), 173–200; MR **12**, 661], essentially by reduction to the propositional case.—The paper contains incidentally several aperçus concerning PC: (i) In paragraph 3, the author says that 'life is easy' as long as one considers all models, without 'imposing boundary conditions'; i.e., the set of formulae of PC which are valid in all models is recursively enumerable, while, if the domain of individuals and the interpretation of one of the relation or function constants are prescribed (e.g., the successor symbol in ω-models of Grzegorczyk, Mostowski and Ryll-Nardzewski [J. Symb. Logic **23** (1958), 188–206; MR **21** #4908]), it need not be (the author inadvertently speaks of the 'size of the models considered', which has nothing to do with the case). (ii) In paragraph 11, 'logical truth is more intimately connected with material falsity than with material truth'; i.e., there are very simple rules of refutability in finite (= 'material') domains, but not for truth; this had been clearly recognized by Herbrand and

emphasized by Hilbert–Bernays both in their introduction and in a whole chapter (pp. 163–178) devoted to the matter, especially pp. 177–178, and expressly called 'criteria of refutability in PC'. The reviewer's no-counter-example-interpretation [J. Symb. Logic **16** (1951), 241–267; **17** (1952), 43–58; MR **14**, 122, 440], which applies also to arithmetic, was given its name to describe the equivalence in classical logic between logical validity and breakdown of all attempts at refutation (by means of a counter example). (iii) In paragraph 12, 'the present approach' (and, presumably other cut-free formalizations) 'recommends itself on grounds of logical technology'; this means not only that proofs in the PC are shortened (Gentzen's 'umweglose Beweise') but that most useful metamathematical results are established easily, e.g., for formulae A in the notation of Hilbert–Bernays' classical number theory Z we have $\vdash_Z[(\vdash_{PC} A)\rightarrow A]$ in Kreisel and Wang [Fund. Math. **42** (1955), 101–110; MR **17**, 447; p. 106], on the basis of a cut-free formalization. As to applied PC one must remember that from a set of axioms $A_1, A_2, \cdots$ one cannot derive all consequences C by cut-free rules; in particular, from the axioms of classical number theory Z of Hilbert–Bernays one cannot derive in a cut-free manner even all quantifier-free consequences. Instead one must define C to be a formal consequence if, for some n, $(A_1 \,\&\, \cdots \,\&\, A_n)\rightarrow C$ can be proved in a cut-free manner; the reviewer does not know any case where this reformulation is troublesome. *G. Kreisel* (Reading)

4634:

Umezawa, Toshio. On intermediate propositional logics. J. Symb. Logic **24** (1959), 20–36.

This is a continuation of the author's previous work [Nagoya Math. J. **9** (1955), 181–189; MR **17**, 446] investigating propositional logics lying between the classical and the intuitionistic.

P. C. Gilmore (Yorktown Heights, N.Y.)

4635:

Farah, Edison. A new definition of ordinal number. Bol. Soc. Mat. São Paulo **12** (1957), 63–69 (1960).

The author provides a definition of "ordinal number" which does not refer to the notion of order, and proves that his definition is equivalent to that of A. Robinson [see Bernays and Fraenkel, *Axiomatic set theory*, North-Holland, Amsterdam, 1958; MR **21** #4912; p. 80], which shares this property. The author claims that his definition offers some advantage in establishing the first basic results for the ordinal numbers, and proves four classical basic results using his definition.

E. J. Cogan (Bronxville, N.Y.)

4636:

Porte, Jean. Systèmes de Post, algorithmes de Markov. Cybernetica **1** (1958), 114–149.

This article describes four familiar formalizations of the intuitive notion of effective process—recursive functions, Post systems, Turing Machines and Markov algorithms. Connections among these formalisms are explained by considering them as particular logistic systems.

The main results are well known but the unified point of view has novel features. The article is intended as an introduction to effective procedures for the non-logician, or perhaps even non-mathematician. While the reviewer lacks first-hand knowledge of Markov's papers, he does feel fairly certain that the author has fulfilled his expository intent.

In the author's account the logistic system $\mathbf{S} = \langle \mathbf{E}, \mathbf{F}, \mathbf{A}, \mathbf{R}, \mathbf{N} \rangle$ consists of the following: **E**, the primitive symbols of S. (**E** can be any finite or denumerable set whatever.) **F**, the well-formed formulas of S. (**F** is an arbitrary subset of the set of finite sequences of elements of **E**.) **A**, the axioms of S. (**A** is an arbitrary subset of **F**.) **R**, the rules of deduction of S. (**R** is a set of relations defined on **F**; **R** is arbitrary except that each member of **R** has at least two terms.) **N**, an enumeration of **E**.

Now let $x, y, z, x_1, \cdots$ be the variables for well-formed formulas in any logistic system S. The following are defined in the obvious way. Definition 4: A deduction in S from $x_1, x_2, \cdots, x_n$. (This is a sequence, possibly infinite, of well-formed formulas of S satisfying a certain condition about $\mathbf{A}, x_1, x_2, \cdots, x_n$, and **R**.) Definition 5: y is deducible from $x_1, x_2, \cdots, x_n$ in S. Definition 6: y is a theorem of S.

In Definition 11 the author defines derivational system. In any logistic system $\mathrm{S}^0 = \langle \mathbf{E}, \mathbf{F}, \mathbf{A}, \mathbf{R}, \mathbf{N} \rangle$, a subset of the deductions in S^0 from $x_1, x_2, \cdots, x_n$ are designated to be derivations in S^0 from $x_1, x_2, \cdots, x_n$. This specification of the derivations is arbitrary except for the condition that if y is a theorem of S^0 then there is a derivation from the empty set of well-formed formulas whose last term is y. With a fixed stipulation of derivations thus associated, S^0 is called a derivational system, say S. The derivational system S is an algorithm if: (1) **A** is empty. (2) Each relation of **R** is binary. (We let "$I(z, y)$" be an abbreviation for "There is a relation R of **R** such that $R(z, y)$".) (3) For any x of **F** there is exactly one derivation from x in S. (Let $D_x = z_1, z_2, \cdots$ be this derivation.) (4) D_x satisfies the following conditions: (4a) z_1 is x; (4b) for any z_i of D_x, z_i not the last term, $I(z_i, y)$ if and only if y is z_{i+1}; if D_x has a last term, say z_N, then for no y of **F** have we $I(z_N, y)$. (For brevity, the definition just given is a shortened, sense-preserving reworking of the author's.)

The word "algorithm" ("algorithme") can be misleading. Clearly there are algorithms in the sense of the author which are non-effective processes, for no demands regarding effectiveness were made in the definitions of logistic system, derivational system, or algorithm. To add such requirements of effectiveness to these definitions and then substitute "an algorithm" for "recursive" in Church's Thesis that every effective process is recursive results in a circular statement. The reviewer is concerned that the reader will somehow jump to the conclusion that the author is suggesting, say on the basis of some unconventional philosophic position, that this circular statement can be used in place of Church's Thesis. This is definitely not the case! (Of course, if we add requirements of recursiveness to the chain of definitions specifying an algorithm, every algorithm becomes a recursive process—a fact which is intuitive evidence for Church's Thesis.) The author's point is simply this: that recursive functions, Post systems, Turing Machines, and Markov algorithms can fruitfully be regarded as instances of logistic systems or derivational systems or algorithms in his sense wherein, in fact, all aspects of these structures are effective.

{Regarding the discussion leading up to Theorem 19, some mention of Thue systems and Post's paper in J. Symb. Logic **12** (1947), 1–11 [MR 8, 558] would seem to have been in order. Post's corresponding development in that article seems to the reviewer to be preferable in

many ways to subsequent variants. The reference of page 130, line −3 should be to Theorem 14, the obvious fact that a recursively enumerable set is effectively enumerable, and the obvious converse of Church's Thesis. The paper contains very many misprints and some vague spots, but most are harmless.}

W. W. Boone (Urbana, Ill.)

4637:

Kreisel, G. A remark on free choice sequences and the topological completeness proofs. J. Symb. Logic **23** (1958), 369–388.

This paper begins with a formalization FC of the theory of free choice sequences. Among the axioms are Brouwer's fan theorem and an axiom stating that if a property $\mathfrak{A}$ holds for a sequence α chosen from a spread G, then, for some p, $\mathfrak{A}$ holds for all sequences β chosen from G whose first p values coincide with those of α. The latter axiom restricts the theory to free choice sequences, as it is false for definite functions as elements of G. On the basis of FC, Tarski's topological completeness proof for the intuitionistic propositional calculus PI [Fund. Math. **31** (1938), 103–134] is developed into an intuitionistic proof of strong completeness for PI [Kreisel, same J. **23** (1958), 317–330; MR **21** #2591]. To this effect the author introduces in the binary spread G_B of sequences of 0, 1 a topology with the finite sequences as the basis of open sets. By means of a numbering of the finite sequences every predicate $\mathfrak{A}(n)$ defines a species of neighbourhoods; the corresponding species of free choice sequences is denoted by A. It is shown that, if a formula $\mathfrak{F}$ is built up by means of propositional connectives from atomic formulas $\alpha \in A_i$, then $\mathfrak{F}$ is equivalent to $\alpha \in E$, where the corresponding $\mathfrak{E}(n)$ contains no variable for free choice sequences. Let $\mathfrak{A}(P_1, \cdots, P_k)$ be a formula in PI and $\Phi_{\mathfrak{A}}(\pi_1, \cdots, \pi_k)$ the function of open sets associated to $\mathfrak{A}$ by Tarski's method. Then

$$\mathfrak{A}(\alpha \in \pi_1, \cdots, \alpha \in \pi_k) \leftrightarrow \alpha \in \Phi_{\mathfrak{A}}(\pi_1, \cdots, \pi_k),$$

where the π_i range over open subsets in G_B, holds in FC. (The corresponding statement in the classical case is false.)

Using a remark of Scott on the recursiveness of certain sets and points used in Tarski's proof, the author proves the following completeness result: If $(\alpha)\mathfrak{A}(\alpha \in \pi_1, \cdots, \alpha \in \pi_k)$ is provable in FC, then $\mathfrak{A}(P_1, \cdots, P_k)$ is provable in PI. The case of prenex formulas in the intuitionistic predicate calculus is reduced to that of PI by the method of Rasiowa and Sikorski [Fund. Math. **41** (1954), 21–28; MR **16**, 987]. It is shown that these results can also be obtained by Beth's methods [Meded. Kon. Nederl. Akad. Wetensch. Afd. Letterkunde (N.R.) **19** (1956), no. 11; MR **19**, 625]. The method of Gödel's completeness proof for the classical predicate calculus leads to a weaker result.

A. Heyting (Amsterdam)

4638:

Kondô, Motokiti. Sur les ensembles nommables et le fondement de l'analyse mathématique. I. Japan. J. Math. **28** (1958), 116 pp.

This paper develops the author's notes in C. R. Acad. Sci. Paris **242** (1956), 1841–1843, 1945–1948, 2084–2087 [MR **17**, 933]. It provides a variant to the arithmetic (=elementary) and analytic (=projective) hierarchies of sets of p-tuples of natural numbers and number-theoretic functions summarized, e.g., by Kleene [Bull. Amer. Math. Soc. **61** (1955), 193–213; MR **17**, 4]. The main formal difference is that the bottom of the author's hierarchy consists of Boolean combinations of polynomial inequalities $p(z_1, \cdots, z_n, \xi_1, \cdots, \xi_k) \geqq 0$, while Kleene has primitive recursive predicates $A(z_1, \cdots, z_n, \alpha_1, \cdots, \alpha_k)$ where the z are individual and α function variables. The difference in interpretation is that the author interprets (as does Kleene) the individual variables as variables over the set N of natural numbers, but allows the coefficients of p to belong to any subfield $\mathbf{k}_0$ (arbitrary, but fixed) of the real numbers, and interprets ξ as ranging over a subfield $\mathbf{k}$, $\mathbf{k} \supset \mathbf{k}_0$, of the real numbers, while Kleene lets α range over N^N itself. But it is clear that many of Kleene's results hold for α ranging over $N_1 \subset N^N$, provided N_1 is closed under suitable operations. Further, since N^N can be mapped very simply on the irrationals, and also primitive recursive relations are arithmetically definable from polynomials, it is to be expected that many of the known results on hierarchies carry over to the author's variant, at least as long as $\mathbf{k}_0$ satisfies suitable closure conditions. The author verifies this in detail. For the reader familiar with the known hierarchies this does not seem to be rewarding, but it may be to a mathematician who is more familiar with polynomials than primitive recursive functionals.

About the first 100 pages are devoted to the description of this hierarchy and its application to the operations of elementary analysis (e.a.).

This work is related to that of Grzegorczyk [Fund. Math. **41** (1955), 311–338; MR **16**, 891] who verified that many theorems of e.a. hold when the (sequences of) real numbers occurring in these theorems are restricted to arithmetically definable ones; to a lesser extent this applies also to arithmetically definable sets. The author goes further by showing that this is largely true uniformly in $\mathbf{k}_0$ and $\mathbf{k}$ provided these fields are closed under arithmetic operations; though the methods of proof are similar to those of Grzegorczyk, which in turn follow standard practice, the latter stated the results only for the smallest field closed under arithmetic operations. It may be observed that the uniformity may also be expected on axiomatic grounds since in e.a. the principle of the least upper bound is in effect only applied to sequences and not to sets; this form of the principle follows from the comprehension principle for subsets P of N in the form $(EP)(n)[n \in P \leftrightarrow R(n)]$ where $R(n)$ is arithmetic and this principle is satisfied by any class of sets closed under (hyper)arithmetic operations.

The last 10 pages consider analytic ($\mathbf{k}_0$, $\mathbf{k}$) hierarchies of sets $\langle \zeta_1, \cdots, q_1, \cdots \rangle$ defined by $(\pm\xi_1)(\pm\xi_2)\cdots(\pm z_1)(\pm z_2)A(\xi_1, \cdots, \zeta_1, \cdots, z_1, \cdots, q_1, \cdots)$, where the ζ are free, ξ bound variables over $\mathbf{k}$, q free, z bound variables over N, A a Boolean combination of polynomial inequalities in ξ, ζ, z, q with coefficients in $\mathbf{k}_0$, and $(+\ )$ denotes the universal quantifier, $(-\ )$ the existential quantifier. The author observes that a numerical quantifier in front of a real number quantifier cannot always be absorbed when $\mathbf{k}$ is arbitrary, since, e.g., even if $(z)(E\xi)_{\mathbf{k}}\cdots$ there need not be a single ξ of $\mathbf{k}$ which represents such a sequence ξ_z; on the other hand, when $\mathbf{k}$ consists of all real numbers, such absorption is of course possible and often used, e.g., Kleene, Trans. Amer. Math. Soc. **79** (1955), 312–340 [MR **17**, 4], p. 315, §§ 3, 4.

The long proof of Theorem 5 (pp. 102–108), which shows that $(E\xi_1)\cdots(E\xi_n)(Ez_1)\cdots(Ez_m)A(\xi_1,\cdots,\xi_n,z_1,\cdots,z_m) \leftrightarrow (Ez_1)\cdots(Ez_p)B_A(z_1,\cdots,z_p)$, when $\mathbf{k}_0$ and $\mathbf{k}$ are closed under arithmetic operations, is not needed, since the theorem is an immediate consequence of Sturm's theorem together with the fact that a real field is real closed if it is closed under arithmetic operations. For, by Sturm's theorem as extended by Tarski, to each A there is a Boolean combination of polynomial inequalities such that $(E\xi_1)_{\mathbf{k}}\cdots A(\xi_1,\cdots,z_1,\cdots)\leftrightarrow B_A(z_1,\cdots)$. (The reviewer has not checked whether the author's proof has any inherent advantages.)

While the paper seems to be a useful summary of results on elementarily definable real numbers, the section on arithmetically definable subsets of $\mathbf{k}$ (pp. 78–85) is hardly adequate. The author does not mention the fact that there are elementarily definable non-empty subsets of the set of all real numbers which do not contain any hyperarithmetic real numbers. A fortiori it is clear that the question whether such a set is open or not is ill-determined unless very strong restrictions on $\mathbf{k}$ are made. On the positive side, the author fails to mention that the theory of measure carries over to arithmetically definable sets provided, as is natural, one considers in the definition of measure coverings by sequences of open intervals $]a_n, b_n[$, where the sequences a_n and b_n are each represented by an element of $\mathbf{k}$. (This definition reduces to the usual one if $\mathbf{k}$ is the set of all real numbers.) Further, e.g., if $\mathbf{k}_0$ is the set of rationals and $\mathbf{k}$ is closed under arithmetic operations, the measure of an arithmetically definable set is itself an arithmetically definable real number.

G. Kreisel (Reading)

4639:

Rabin, Michael O. Computable algebra, general theory and theory of computable fields. Trans. Amer. Math. Soc. **95** (1960), 341–360.

The author's own introduction, only slightly modified, seems to summarize this attractive paper perfectly. In the passage from such concrete systems as the ring of integers or the field of rationals to the corresponding abstract systems various properties are lost such as, in the case of the rationals, (a) a natural topology, (b) an ordering, (c) the effective computability of $+$ and $\times$, (d) their continuity with respect to (a). The properties (a)–(d) do not enter into the axiomatic concept of a field at all. Some of them are reincorporated in later developments, e.g., in topological algebra one requires a topology satisfying (d) and studies primarily homomorphisms which are continuous mappings. The present paper is concerned with (c) and considers a denumerable algebraic system such that there is at least one indexing of its objects (by natural numbers) which makes the algebraic operations computable. Here computable = recursive, though most positive theorems hold if computable is taken to mean: belonging to any fixed class closed under recursive operations. In the case of groups the author shows how to introduce, given a computable group, an indexing for factor groups and subgroups which makes the natural homomorphisms and isomorphisms computable. A finitely generated, though not necessarily finitely presented, group has a solvable word problem if and only if it is computable. Hence solvability of the word problem is invariant for any change from one finite set of generators to another. An interesting application is that any finitely generated group which has a faithful matrix representation is necessarily recursive (i.e., computable in the strongest sense considered here). This not only gives a new proof of a result by Fuchs-Rabinowitsch [Dokl. Akad. Nauk. SSSR **27** (1940), 425–426; **29** (1940), 549–550; MR **2**, 126, 307] but shows that the notion of a recursive group is useful for answering a question which does not refer to this notion, namely, what finitely generated groups have a faithful matrix representation? The main result on computable fields F provides a computable indexing for the algebraic closure $\bar{F}$ such that the natural isomorphic embedding of F in $\bar{F}$ is computable. Here the notion of a computable field allows a precise distinction to be made between (i) Steinitz's classical construction of $\bar{F}$ and (ii) Bourbaki's [*Algèbre, Chap. IV, V*, Hermann, Paris, 1950; MR **12**, 6]. For, (i) uses the factorization of polynomials over intermediate fields which is not valid in computable algebra since there is a computable F for which these intermediate fields do not have a factorization algorithm, while (ii) is valid. On the other hand it seems that on the basis of the standard axiomatic approach, a preference for (ii) over (i) would have to be considered 'extra-mathematical'.—It seems that algebra (= calculating) is much better suited to an analysis by means of recursion theory than those branches of mathematics which are more closely related to geometric conceptions.

G. Kreisel (Reading)

COMBINATORIAL ANALYSIS

See also 4732, B5145, B5532.

4640:

Klein-Barmen, Fritz. Über eine bei der Zerlegung einer endlichen Menge auftretende elementare zahlentheoretische Funktion. Jber. Deutsch. Math. Verein. **62**, Abt. 1, 130–134 (1960).

Let N be a set of r elements and denote by $\zeta(r, s)$ the number of partitions of N into s pairwise disjoint non-empty subsets. A difference equation satisfied by $\zeta(r, s)$ is found; and induction with respect to the two arguments r and s is then used to establish the identity

$$(s-1)!\zeta(r,s) = \sum_{\sigma=0}^{s-1}(-1)^{\sigma}\binom{s-1}{\sigma}(s-\sigma)^{r-1}.$$

L. Mirsky (Sheffield)

4641:

Masuyama, Motosaburo. On the existence theorem of the cyclic difference sets for the orthogonal arrays $(9\lambda, 3\lambda+1, 3, 2)$. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **7** (1960), 53–55.

Let $A, B, \cdots$ denote sets of elements (with repetitions) of a finite Abelian group. Write $A+B$ for the union (repetitions counted) of A and B, $A-B=C$ if $A=B+C$. If $A=\{a_i\}$, $B=\{b_j\}$ set $A\cdot B=\{a_i+b_j\}$, $A^*=\{-a_i\}$, $E=\{0\}$. Let further L be the empty set and λA the sum of λ summands A.

The author has previously [same Rep. **6** (1959), 47–53; MR **22** #1049] devised a method to construct orthogonal arrays $(9\lambda, 3\lambda+1, 3, 2)$ from certain partitions of the residues mod $3\lambda-1$ into 3 sets A, B, C, when $3\lambda-1$ is a

prime. The sets A, B, C must satisfy the conditions: (1) $BC^* = AA^* - \lambda E$; (2) $AC^* = BB^* - \lambda E$; (3) $AB^* \equiv CC^* - \lambda E \pmod R$, where R is the set of all residues mod $3\lambda - 1$. In the present paper he shows: If sets A, B, C satisfy any one of the three conditions together with the condition $A + B + C \equiv L \pmod R$, then an orthogonal array $(9\lambda, 3\lambda + 1, 3, 2)$ can be constructed.

H. B. Mann (Columbus, Ohio)

4642:

Nash-Williams, C. St. J. A. Decomposition of graphs into closed and endless chains. Proc. London Math. Soc. (3) **10** (1960), 221–238.

The author presents a generalization of the theory of unicursal paths which applies to all graphs, finite or infinite. The "chains" of the title are graphs having a unicursal path (which traverses each edge just once). They are closed if the path is reentrant and "endless" if the path has no beginning or end.

The following theorems are established. (1) A graph G can be decomposed into closed and endless chains (so that each edge belongs to just one chain) if and only if it has no vertex of odd valency. (2) G is decomposable into endless chains if and only if it has no vertex of odd valency and no finite component. (3) G is decomposable into closed chains if and only if it has no odd cincture (cut).

The author's three theorems and 29 lemmas provide an excellent introduction to the theory of infinite graphs.

W. T. Tutte (Toronto)

4643:

Dirac, Gábor. Généralisations du théorème de Menger. C. R. Acad. Sci. Paris **250** (1960), 4252–4253.

The well-known theorem of Menger is generalized to infinite graphs G by proving: If points a and b are not separated in G by any set S of points of cardinality $|S| < k$, then G contains a family C of chains having only points a, b in common such that $|C| \geq k$.

F. Harary (Ann Arbor, Mich.)

4644:

Dirac, Gábor. Un théorème de réduction. C. R. Acad. Sci. Paris **251** (1960), 24–25.

Let G be a graph of chromatic number k such that no graph of chromatic number $\geq k$ can be obtained from it by contracting one or more edges (each to a single vertex). The author shows that for $k \geq 5$ G is 5-connected.

W. T. Tutte (Toronto)

4645:

Camion, Paul. Quelques propriétés des chemins et circuits hamiltoniens dans la théorie des graphes. Cahiers Centre Études Rech. Oper. **2**, 5–36 (1960).

The graphs considered in this paper are oriented and "complete". The latter term implies that any two distinct vertices are joined by at least one edge. The terms "Hamiltonian path" and "Hamiltonian circuit" are used with the understanding that the orientation of the graph is to induce a consistent orientation in the path or circuit under discussion. The author gives necessary and sufficient conditions for existence and for uniqueness of Hamiltonian paths and circuits subject to various specified restrictions.

W. T. Tutte (Toronto)

4646:

Tutte, W. T. A non-Hamiltonian graph. Canad. Math. Bull. **3** (1960), 1–5.

The following cubic graph having 28 vertices and containing no circuit with fewer than 7 edges was discovered by H. S. M. Coxeter: the graph contains three mutually disjoint heptagons $a_1, a_2, a_3, a_4, a_5, a_6, a_7$; $b_1, b_5, b_2, b_6, b_3, b_7, b_4$; $c_1, c_6, c_4, c_2, c_7, c_5, c_3$; and seven more vertices $d_1, \cdots, d_7$, d_i being joined to a_i, b_i and c_i for $i = 1, \cdots, 7$. In this paper it is proved that this graph has no Hamiltonian circuit. The proof utilises the strong symmetry properties of the graph.

G. A. Dirac (Hamburg)

4647:

Dantzig, George B. On the shortest route through a network. Management Sci. **6** (1959/60), 187–190.

This paper gives a slightly modified formulation of an algorithm (of which more than six other formulations have been recently published by various authors) for finding the shortest path through a maze. This formulation is claimed to be faster than others, if the assumption is made (somewhat unrealistically, if the algorithm is to be performed by a computing machine) that the time required to sort numbers is negligible compared to the time required to add them.

E. F. Moore (Murray Hill, N.Y.)

ORDER, LATTICES

See also B5512, B5513, B5514, B5515, B5516, B5519, B5520, B5521, B5522, B5527.

4648:

Crawley, Peter. Lattices whose congruences form a boolean algebra. Pacific J. Math. **10** (1960), 787–795.

Let L be a lattice, $\Theta(L)$ its lattice of congruence relations. In the paper certain connections between L and $\Theta(L)$ are studied. The author's investigation is based upon his lemma that the lattice $\Theta(L)$ is compactly generated [Bull. Amer. Math. Soc. **65** (1959), 377–379; MR **21** #5590]. In the first section of the paper the author deals with the sequence $L \leq \Theta(L) \leq \Theta(\Theta(L)) \leq \cdots$ of iterated congruence lattices of a distributive lattice L, and proves the following theorem: If D is a distributive lattice of infinite cardinality α, then every distributive lattice L of cardinality $\leq 2^\alpha$ can be embedded in $\Theta^4(L) = \Theta(\Theta(\Theta(\Theta(L))))$ (but in general cannot be embedded in $\Theta(L)$; for the case $\Theta^2(L)$ the question is open). Corollaries: 1. An arbitrary lattice L is isomorphic with $\Theta(L)$ if and only if L is a finite Boolean algebra. 2. For a distributive lattice D, the chain $D \leq \Theta(D) \leq \Theta^2(D) \leq \cdots$ terminates if and only if D is finite. In the second section the author gives a new characterisation of lattices whose congruence lattices form a Boolean algebra. The author's characterisation is based upon the concept of minimal quotient. A quotient a/b, $a \neq b$, is called minimal if for each quotient c/d, $c \neq d$, weakly projective into a/b, there exists a finite chain $a = x_0 > x_1 > \cdots > x_k = b$ such that x_{i-1}/x_i is weakly projective into c/d ($i = 1, 2, \cdots, k$). Theorems: 1. For an arbitrary lattice L, the congruence lattice $\Theta(L)$ is a Boolean algebra if and only if, for each proper quotient a/b of L, there exists a finite chain $a = c_0 > c_1 > \cdots > c_r = b$ such that each c_{i-1}/c_i is minimal. 2. Let L be a modular lattice each

proper quotient of which contains a prime quotient. Then $\Theta(L)$ is a Boolean algebra if and only if each quotient of L is finite dimensional. *V. Vilhelm* (Prague)

4649:

Finkbeiner, D. T. Irreducible congruence relations on lattices. Pacific J. Math. **10** (1960), 813–821.

By 'δ-lattice' the author denotes a lattice L which satisfies the descending chain condition and each join-irreducible element of which covers some element of L. By means of weak projectivity of prime quotients the relationship between a δ-lattice L and the lattice $\Theta(L)$ of all congruence relations on L is studied. Definitions: Let q be a join-irreducible element of a δ-lattice L and $q \succ c_q$. The minimal congruence relation which collapses the quotient q/c_q is called the irreducible congruence relation ϑ_q. Let ω be the null element of $\Theta(L)$. L is called simple, if the congruence relations on L are trivial; L is irreducible if there exist elements $a \neq b$ such that $a \equiv b\ (\vartheta)$ for every $\vartheta \neq \omega$. The author shows that the irreducible congruence relations ϑ_q on a δ-lattice L generate $\Theta(L)$, and proves the following theorems. 1. $\Theta(L)$ is a Boolean algebra if one of the following conditions is satisfied: (a) for every join-irreducible element q, $\vartheta_q \succ \omega$; (b) the relation of weak projectivity is symmetric on the set of all irreducible prime quotients. 2. Let each two irreducible mutually weakly projective prime quotients be projective. Then L is simple if and only if all prime quotients are projective. 3. If $\Theta(L)$ is a Boolean algebra, then L is simple if and only if L is irreducible. 4. $\Theta(L)$ is a Boolean algebra if and only if ϑ_q has a complement for every join-irreducible element $q \in L$. {Reviewer's remark: The statement (b) of the theorem 1 is contained in a result of J. Jakubík [Časopis Pěst. Mat. **80** (1955), 206–216; MR **17**, 1177].}

V. Vilhelm (Prague)

4650:

Aubert, Karl Egil. Un théorème d'immersion pour une classe étendue de structures algébriques réticulées. An. Acad. Brasil. Ci. **31** (1959), 321–329.

A c-algebra is defined to be an algebraic structure A satisfying the axioms for a lattice-ordered algebra (over a partially ordered ring Ω) except possibly the commutativity of addition and the commutativity and associativity of multiplication. Using the results of his theory of x-ideals [*Séminaire Dubreil-Pisot 1955/56*, Paris, 1956; MR **19**, 522], the author shows that a c-algebra can be embedded in a direct product of simply ordered c-algebras if and only if it satisfies the condition: $a \wedge b = 0$ implies $a \wedge |cb| = a \wedge |bc| = a \wedge |\alpha b| = a \wedge |c+b-c| = 0$ for all $a, b, c \in A$, $\alpha \in \Omega$. *R. S. Pierce* (Seattle, Wash.)

4651:

Hukuhara, Masuo. ⊕-endomorphisme et ∩-endomorphisme d'un treillis en dualité et la théorie de Riesz sur l'endomorphisme complètement continu. III. Applications aux endomorphismes d'un espace vectoriel. Funkcial. Ekvac. **2** (1959), 19–32. (Esperanto summary)

[For parts I and II see Funkcial. Ekvac. **1** (1958), 85–102, 103–120; MR **20** #3083, #6375.] The present chapter in the series inspects certain aspects of spectral theory from the lattice-theoretic point of view.

R. M. Baer (Berkeley, Calif.)

GENERAL MATHEMATICAL SYSTEMS

See also 4639.

4652:

Boccioni, Domenico. Condizioni di distributività ed associatività unilaterali. Rend. Sem. Mat. Univ. Padova **30** (1960), 178–193.

The $2\nu^3$ conditions for left-distributivity and additive associativity of an abstract algebraic system of order ν are independent if $\nu \geqq 4$. The same applies to multiplicative associativity. The cases $\nu < 4$ are solved in detail.

H. A. Thurston (Vancouver, B.C.)

THEORY OF NUMBERS

See also 4640, 4706, 4708, 4740.

4653:

Vandiver, H. S. On the use of abstract algebra as a method for obtaining results involving rational integers only and the reverse of this procedure. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 545–554.

The author says: "The present paper constitutes only a beginning in our investigation of the relations between abstract algebra and number theory.... Our principal aim in pursuing the present investigation is to isolate, if possible, certain tools in abstract algebra or the theory of algebraic numbers which seem to have been the most successful in deriving results concerning the rational integers alone."

For that purpose, the author gives in the present paper a historical survey of the application of algebraic methods in number theory developed since Lagrange and Gauss; he presents an interesting brief history of algebraic number theory. *K. Iwasawa* (Cambridge, Mass.)

4654:

Khatri, M. N. Triangular numbers which are also squares. Math. Student **27** (1959), 55–56.

4655:

Świerczkowski, S. On Pythagorean angles. Colloq. Math. **7** (1959), 103–105.

An angle α is called Pythagorean if it is congruent modulo $\pi/2$ to an angle of a right triangle, the lengths of whose sides are integers. The author proves that the Pythagorean angles are precisely those that are sums (modulo $\pi/2$) of the Pythagorean angles that arise from primes of the form $p = 4k+1$ in the well-known way (i.e., the uniquely determined Pythagorean angles which correspond to triangles whose hypotenuse is p).

L. A. Rubel (Urbana, Ill.)

4656:

Battaglia, Antonio. Formule parametriche per triangoli eroniani. Archimede **11** (1959), 163–167.

Using the formula of Heron (more correctly, of Archimedes), $S^2 = p(p-a)(p-b)(p-c)$, for the area of a triangle having a, b, c for sides, the author derives the formulas

$$(*) \qquad \begin{aligned} a &= \lambda^2 + 9\mu^2, \quad & b &= 2(\lambda^2 + 3\mu^2), \\ c &= 3(\lambda^2 + \mu^2), \quad & S &= 6(\lambda^2 + \mu^2)\lambda\mu, \end{aligned}$$

which, for integral values of λ, μ give integers for a, b, c, and S (i.e., a Heronian triangle, by definition); furthermore, a, b, c form an arithmetic progression. By expressing S in terms of two sides and the sine of the included angle the author obtains another set of formulas having the same properties as (*).

N. A. Court (Norman, Okla.)

4657:

Selfridge, J. L.; Straus, E. G. On the determination of numbers by their sums of a fixed order. Pacific J. Math. 8 (1958), 847–856.

Let $\{x\}=\{x_1, x_2, \cdots, x_n\}$ be a set of complex numbers and let $\{\sigma\}=\{\sigma_1, \sigma_2, \cdots, \sigma_{\binom{n}{s}}\}$ be the set of sums of s disjoint elements of $\{x\}$. The authors consider to what extent $\{x\}$ is determined by $\{\sigma\}$ and what sets can be $\{\sigma\}$ sets. In the case $s=2$ manipulations with symmetric functions of the x's and σ's reveal that if $n\neq 2^k$ then $\{x\}$ is uniquely determined by $\{\sigma\}$. If $n=2^k$ there exist at least two sets $\{x\}$ which have the same $\{\sigma\}$. It is conjectured that if $k\geq 2$ there cannot be three sets $\{x\}$ giving rise to the same $\{\sigma\}$, but in a recent letter to the reviewer the second author has given a counter-example to this conjecture for the case $k=3$. For every s a system of Diophantine equations $f(n, k)=0$ $(k=1, 2, \cdots, n)$ is obtained such that if n satisfies none of these, the first n elementary symmetric functions of $\{\sigma\}$ can be prescribed arbitrarily and determine the $\{x\}$ uniquely. If $f(n, k)=0$, then the first k elementary symmetric functions of $\{\sigma\}$ satisfy an algebraic equation. In the case $s=3$ the equation is $f(n, k)=n^2-(2^k+1)n+2\cdot 3^{k-1}=0$. This is shown to have only the solution pairs $(n, k)=(1, 1)$, $(2, 1)$, $(3, 2)$, $(3, 3)$, $(6, 3)$, $(6, 5)$, $(27, 5)$, $(27, 9)$, $(486, 9)$. It is deduced that for $s=3$ and $n\neq 1, 2, 3, 6, 27, 486$, $\{x\}$ is determined by $\{\sigma\}$. However, for the cases $n=27, 486$ the question of whether $\{\sigma\}$ always determines $\{x\}$ is left open. Similar results are obtained for other values of s. It is also shown that if $n>s$, and if there exist nontrivial transformations $y_i=f_i(x_1, x_2, \cdots, x_n)$ which preserve $\{\sigma\}$, then $n=2s$, the transformations must be linear, and their form is determined.

L. Moser (Edmonton, Alta.)

4658:

Veržbinskiĭ, M. L. The primeness deficiency of an integer. Zap. Leningrad. Gorn. Inst. **36** (1958), no. 3, 5–12. (Russian)

The primeness deficiency of the natural number n is defined to be $\kappa(n)=\sum_1^{n-1}(m, n)-(n-1)$, where (m, n) is the greatest common divisor. Then it is shown that

$$(1)\qquad \kappa(n) < n^{1+[\log\log(4\log n)-\log\log 2]/\log\log n}-2n+1$$

"asymptotically" for all large n. Clearly $\kappa(n)$ is zero only when n is prime, and the author shows that

$$\kappa(n) > (n^{1/2}-1)^2$$

when n is composite. He also obtains bounds below for $\kappa(n)$ depending on the number of prime factors of n. The proofs are all elementary. {It appears to the reviewer that "asymptotically" in connection with (1) means that the inequality with $(1+\varepsilon)$ multiplying the log log $(4\log n)$ term is true for all sufficiently large n.}

J. W. S. Cassels (Cambridge, England)

4659:

Wang, Yuan. On the least primitive root of a prime. Acta Math. Sinica **9** (1959), 432–441. (Chinese)

Let p be a prime, $g(p)$ its smallest primitive root; let further $w(n)$ be the number of distinct prime factors of n, and $m=w(p-1)$. The author proves that $g(p)=O(p^{1+\varepsilon})$ $(\varepsilon>0)$, and that, if the Riemann hypothesis holds, even $g(p)=O(m^6(\log p)^2)$. Both results are important improvements on what was known. The proofs of these theorems are based on the following two theorems: (A) Let $0<\delta\leq\frac{1}{4}$. If $p>P(\delta)$ and $H>p^{1/4+\delta}$, then for every positive integer N we have $|\sum_{n=N+1}^{N+H}\chi(n)|<H/p^{\eta}$, where $\eta=\delta^2/6$ and χ is a character (mod p) distinct from the principal one; (B) if the Riemann hypothesis holds and χ is not the principal character,

$$\sum_{n=1}^{\infty}\Lambda(n)\chi(n)e^{-n/x}=O(x^{1/2}\log p).$$

K. Mahler (Manchester)

4660:

Satyanarayana, M. A note on Fermat and Mersenne's numbers. Math. Student **26** (1958), 177–178.

Author's introduction: "In this note it is proved that no Fermat number is a triangular number, and also that numbers of the form 2^n-1, where n is an odd integer >1, are not triangular numbers."

4661:

Knopfmacher, J. A note on perfect numbers. Math. Gaz. **44** (1960), 45.

A proof, which "appears to be new", of the theorem that every even perfect number is of the form $2^n(2^{n+1}-1)$, where $2^{n+1}-1$ is prime.

4662:

Gloden, A. ★Table des solutions de la congruence $X^4+1=0 \pmod p$ pour $800000<p<1000000$. Chez l'auteur, Luxembourg, rue Jean Jaurès, 11; 1959. 23 pp. (mimeographed) 150 francs belges.

The two solutions $<\frac{1}{2}p$ of the congruence mentioned in the title are given for each prime p of the form $8x+1$ within the indicated limits. This extends the previously calculated tables of Cunningham [*Binomial factorisations*, 1923–25] and those of Delfeld and the author [for earlier references to tables of the author see MR **7**, 145; **8**, 564; **9**, 270; **15**, 288; **17**, 827; see also Delfeld, Inst. Grand-Ducal Luxembourg. Sect. Sci. Nat. Phys. Math. Arch. (N.S.) **16** (1946), 65–70; MR **8**, 564].

D. H. Lehmer (Berkeley, Calif.)

4663:

Cohen, Eckford. Arithmetical inversion formulas. Canad. J. Math. **12** (1960), 399–409.

An arithmetical function $f(n, r)$ is called even modulo r if $f(n, r)=f((n, r), r)$, where (n, r) denotes the g.c.d. of n and r. The author characterizes the class of even functions (mod r) by showing that they have a unique representation of the form

$$f(n, r)=\sum_{d|(n, r)}F\left(d, \frac{r}{d}\right),$$

where $F(r_1, r_2)=\sum_{s|r_1}f(r_1/d, r_1r_2)\mu(d)$. This generalizes

the familiar Möbius inversion formula. He also proves a companion result for a certain subclass of even functions known as primitive functions mod r. The paper also contains generalizations of a number of arithmetical identities due to Hölder, Landau, Brauer and Rademacher.

T. M. Apostol (Pasadena, Calif.)

4664:

Cohen, E. Arithmetical functions associated with arbitrary sets of integers. Acta Arith. **5**, 407–415, 435 (1959).

For a set S of positive integers n the following arithmetical functions are considered: $[x]_S$ = the number of positive integers $\leqq x$, contained in S; $\tau_S(n)$ = the number of positive divisors of n contained in S; $\varphi_S(n)$ = the number of integers a (mod n) with $(a, n) \in S$; $N_k(x, S)$ = the number of k-dimensional vectors $\{n_1, \cdots, n_k\}$, $1 \leqq n_i \leqq x$, $i = 1, \cdots, k$, with $(n_1, n_2, \cdots, n_k) \in S$; $T_k(x, S) = \sum_{n_i \leqq x; i=1,2,\cdots,k} \tau_S(n_1, \cdots, n_k)$; $\varphi_S(x, n)$ = the number of positive integers $a \leqq x$ such that $(a, n) \in S$. Introducing the characteristic function $\rho_S(n)$ of the set S and a suitable inversion function $\mu_S(n)$ defined in such a way that $\sum_{d|n} \mu_S(d) = \rho_S(n)$, the author gives relations such as

$$T_k(x, S) = \sum_{n \leqq x} \rho_S(n) \left[\frac{x}{n}\right]^k, \quad N_k(x, S) = \sum_{n \leqq x} \mu_S(n) \left[\frac{x}{n}\right]^k,$$

$$\varphi_S(x, n) = \sum_{d|n} \mu_S(d) \left[\frac{x}{d}\right].$$

He also gives asymptotic estimates of $T_k(x, S)$ and $N_k(x, S)$ for large x.

H. Bergström (Göteborg)

4665:

Negoescu, N. C. Sur quelques relations entre des fonctions arithmétiques. Com. Acad. R. P. Romîne **9** (1959), 989–994. (Romanian. Russian and French summaries)

The two following theorems are established. (I) If a correspondence is established between the sets of the positive integers a_i and b_i ($i = 1, 2, \cdots, n$), and if S denotes the sum of the products of the corresponding numbers, S_d the sum of the numbers b_i corresponding to the numbers a_i that are multiples of d, and S_δ the sum of the numbers a_i corresponding to the numbers b_i that are multiples of δ, then $S = \sum_d \phi(d) S_d = \sum_\delta \phi(\delta) S_\delta$, where d is successively equal to all the distinct divisors of the numbers a_i, and δ to all those of the numbers b_i, and where ϕ is Euler's function. A number of relations are derived from this result. (II) If $F(n)$ and $G(n)$ are two product functions and $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the decomposition in prime factors of the number n, then

$$\sum_{d|n} F(d)G(n/d) = \prod_{i=1}^{k} [F(p_i^{\alpha_i}) + F(p_i^{\alpha_i - 1})G(p_i) + F(p_i^{\alpha_i - 2})G(p_i^2) + \cdots + G(p_i^{\alpha_i})].$$

If $n = 1$, the second member is taken equal to unity. A number of consequences are derived from this result; in particular, the inversion formula of Möbius.

E. Frank (Chicago, Ill.)

4666:

Sugunamma, M. Certain results concerning $\sigma_k(n)$ and $\varphi_k(n)$. Ann. Polon. Math. **8** (1960), 173–176.

Typical result is

$$\lim_{n \to \infty} \frac{\sigma_n(n+a)}{\sigma_n(n)} = e^a,$$

where $\sigma_k(n)$ denotes the sum of the kth powers of the divisors of n.

S. Chowla (Notre Dame, Ind.)

4667:

Ciesielski, Z. On multiplicative sequences. Colloq. Math. **7** (1959/60), 265–268.

Let $f(n) = \pm 1$ be a multiplicative number-theoretic function. It has been conjectured that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(k) \tag{1}$$

always exists. Define the real number t ($0 \leqq t \leqq 1$) as follows: $t = \sum_{n=1}^{\infty} (1 + f(p_n))/2^{n+1}$, where p_n is the nth prime. The author proceeds to prove that (1) holds for almost all multiplicative functions. He also proves this for all complex-valued multiplicative functions satisfying $|f(n)| = 1$ (see also A. Wintner).

P. Erdös (Haifa)

4668:

Delange, Hubert. Sur des formules dues à Atle Selberg. Bull. Sci. Math. (2) **83** (1959), 101–111.

Let $\omega(n)$ represent the number of distinct prime factors of the positive integer n. Then a particularization of a theorem of Selberg [J. Indian Math. Soc. (N.S.) **18** (1954), 83–87; MR **16**, 676] yields the following formula:

$$\sum_{n \leqq x} z^{\omega(n)} = x(\log x)^{z-1} F(z) + O(x(\log x)^{z-2}). \tag{1}$$

In (1), x is positive and large, z is an arbitrary complex number, and F denotes the entire function defined by:

$$F(z) = (\Gamma(z))^{-1} \prod_p (1 - 1/p)^z (1 + z/(p-1)),$$

where the infinite product extends over all primes. The O-term in (1) is uniform in z for z bounded in modulus.

With (1) as a starting point, the author derives, or indicates the derivation of, numerous results concerning $\omega(n)$, both old and new. In particular, a short proof leads from (1) to the fact that, for any non-negative integer m,

$$\sum_{n \leqq x} (\omega(n) - \log\log x)^m = m! x A_m(x) + O(x(\log\log x)^{m/2}/\log x),$$

where $A_m(x)$ is the coefficient of ζ^m in the Maclaurin expansion of $F(e^\zeta) \exp((e^\zeta - 1 - \zeta)\log\log x)$. Thus $A_m(x)$ is a polynomial in $\log\log x$ of degree $[m/2]$, where $[x]$ denotes, as usual, the greatest integer function. This estimate is an improvement over a previous one due to the author [C. R. Acad. Sci. Paris **237** (1953), 542–544; MR **15**, 201].

Another result, stated by the author without proof (which is to appear elsewhere) is the following. Let $N(x, t)$ denote the number of positive integers $n \leqq x$ for which $\omega(n) \leqq \log\log x + t(\log\log x)^{1/2}$. Then

$$N(x, t)/x = (2\pi)^{-1/2} \int_{-\infty}^{t} \exp(-u^2/2)du + (2\pi \log\log x)^{-1/2} \exp(-t^2/2) \times (a - t^2/6 - \{\log\log x + t(\log\log x)^{1/2}\}) + O(\log\log x)^{-1},$$

where $\{x\}$ denotes the fractional part of x, and $a = 2/3 - F'(1) = 0.40516\cdots$. This improves upon the results of Rényi and Turán [Acta Arith. 4 (1958), 71–84; MR **20** #3112], who, in demonstrating the validity of a conjecture of LeVeque [Trans. Amer. Math. Soc. **66** (1949), 440–463; MR **11**, 83], showed that

$$N(x,t)/x = (2\pi)^{-1/2}\int_{-\infty}^{t} \exp(-u^2/2)du + O(\log\log x)^{-1/2}.$$

Formulas similar in form to (1) apply when the left-hand side of (1) is replaced by a summation extending only over squarefree n, or when the function $\omega(n)$ is replaced by $\Omega(n)$, the total number of prime factors, distinct or not, of n. Accordingly, results corresponding to those flowing from (1) are to be expected in these latter cases as well; various examples of such corresponding results are provided by the author.

A. Sklar (Chicago, Ill.)

4669:

Vandiver, H. S. On distribution problems involving the numbers of solutions of certain trinomial congruences. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 1635–1641.

The author continues the study of the number of solutions of the congruence $1 + ax^c \equiv by^l \pmod p$, where p is prime, $cl + 1 = p$, and $xy \not\equiv 0 \pmod p$. [See E. H. Pearson and H. S. Vandiver, same Proc. **39** (1953), 1278–1285; MR **15**, 684; Emma Lehmer and H. S. Vandiver, J. Assoc. Comput. Mach. **4** (1957), 505–510; MR **20** #428; and H. S. Vandiver, same Proc. **44** (1958), 459–464; MR **20** #3648.] This congruence is written in the form $1 \equiv g^{l+ls} + g^{l+cr} \pmod p$, where g is a primitive root modulo p. For given p, g, c, and l, let $[i, j]$ denote the number of pairs r, s for which the congruence is satisfied with $0 \leq r < l$ and $0 \leq s < c$. Let l be a fixed odd prime and let $l = b_0 + b_1 + \cdots + b_{l-1}$ be a fixed partition of l into non-negative parts smaller than l. Let t be a fixed rational integer such that neither $t+1$ nor $(t^l+1)/(t+1)$ is an lth power and such that $t + 1 \not\equiv 0 \pmod l$. It is shown that there exist an infinite number of primes p of the form $p = cl + 1$ with $c \not\equiv 0 \pmod l$ such that if g is any primitive root of p and if $t \equiv -g^v \pmod p$, then $[v, 0], [v, 1], \cdots, [v, l-1]$ is the set of b's in some order. *W. H. Mills* (Berkeley, Calif.)

4670:

Schinzel, A.; Sierpiński, W. Sur les congruences $x^x \equiv c \pmod m$ et $a^x \equiv b \pmod p$. Collect. Math. **11** (1959), 153–164.

Theorem 1: If m is a positive integer and $(c, m) = 1$, then the congruence $x^x \equiv c \pmod m$ has an infinite number of solutions x such that $(x, \phi(m)) = 1$. As a corollary, if p is prime the congruence $x^x \equiv c \pmod p$ has an infinite number of solutions. It is noted that a modification of the proof of theorem 1 leads to the following result: If $(c, m) = 1$, then each of the congruences

$$x^x \equiv c \pmod m, \quad x^{x^x} \equiv c \pmod m, \cdots$$

has an infinite number of solutions.

Theorem 2: If $(a, b) = 1$, $|a| > 1$, $b \neq 0$, then there exist infinitely many primes p such that the congruence $a^x \equiv b \pmod p$ has infinitely many solutions. It is noted that by making use of a result of Pólya [Math. Z. **1** (1918), 143–148], it can be shown that if $a > 0$, $b \neq 0$, then for given m there exists an integer $k_0 \geq 1$ such that for all $k > k_0$ the number $a^k - b$ has a prime divisor $> m$.

Theorem 3: If $a \geq 1$ and $b \neq$ the 12th power of an integer, $(a, b) = 1$, then there exists an infinite number of primes p such that the congruence $a^x \equiv b \pmod p$ has no solutions in integers $x = 0, 1, 2, \cdots$. It is remarked that Schinzel has proved that if $a \neq b^k$ then there exist infinitely many primes p such that $a^x \equiv b \pmod p$ has no solution.

L. Carlitz (Durham, N.C.)

4671:

Carlitz, L. Composition of sequences satisfying Kummer's congruences. Collect. Math. **11** (1959), 137–152.

Let p be a fixed prime and let $\{a_n\}$ be a sequence of rational numbers that are integral (mod p) and satisfy the congruence

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} a_{p^{r-s}} a_{m+s(p-1)} \equiv 0 \pmod{p^r}$$

for all $m \geq r \geq 1$. Let $\{b_n\}$ be a second sequence of rational numbers that are integral (mod p) and satisfy the above congruence with a replaced by b. The Hurwitz product $\{c_m\}$ of the sequences $\{a_m\}$, $\{b_m\}$ is defined by means of $c_m = \sum_{j=0}^{m} \binom{m}{j} a_j b_{m-j}$ $(m = 0, 1, 2, \cdots)$. In an earlier paper [Arch. Math. **7** (1957), 441–445; MR **19**, 120] the author has derived properties of $\{c_m\}$ under the hypothesis that $a_p \equiv b_p \pmod p$. In the present paper he considers the general situation in which this congruence is no longer assumed. Two of his main theorems are as follows. (1) If $a_p b_p \not\equiv 0 \pmod p$ and t is the least positive integer such that $a_p{}^t \equiv b_p{}^t \equiv k \pmod p$ for some k, then

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} k^{r-s} c(m + s(p^t - 1)) \equiv 0 \pmod{p^r}$$

for $m \geq r \geq 1$, where c_m has been replaced by $c(m)$. (2) If $a_p \not\equiv 0$ but $b_p \equiv 0 \pmod p$, then

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} a_p{}^{r-s} c(m + sp(p-1)) \equiv 0 \pmod{p^r}$$

for $m \geq rp$. In certain applications, for example, power series expansions of elliptic functions, sequences with polynomial coefficients occur; the above theorems now require some modification. The final portion of the paper contains a number of applications of these results. In particular, congruences are obtained for the coefficients of various combinations of trigonometric, hyperbolic and elliptic functions. *A. L. Whiteman* (Princeton, N.J.)

4672:

Comment, Pierre. Divisibilité faible. Rend. Circ. Mat. Palermo (2) **8** (1959), 333–340.

If b and c are positive integers, then b is called a weak divisor of c if every prime divisor of b divides c (the reviewer suggests the notation $b | c^\infty$). The paper gives a number of definitions and immediate consequences.

N. G. de Bruijn (Eindhoven)

4673a:

Battaglia, Antonio. Sull'equazione indeterminata $x^{2n} + y^{2n} = z^2$. Boll. Un. Mat. Ital. (3) **14** (1959), 498–499.

4673b:

Ricci, Giovanni. Una osservazione sulla precedente Nota "A. Battaglia, Sull'equazione indeterminata $x^{2n}+y^{2n}=z^{2n}$". Boll. Un. Mat. Ital. (3) **14** (1959), 499–500.

In a previous paper [same Boll. (3) **12** (1957), 689–694; MR **19**, 941] Battaglia showed that integers x, y, z satisfying the conditions $x^{2n}+y^{2n}=z^2$, $(x, y, z)=1$, $(n, z)=1$ $(n>1)$, and such that x^2+y^2 is the square of a power of a prime, do not exist. In the present note he attempts to remove the last condition. The validity of his reasoning is questioned in the note by Ricci.

D. H. Lehmer (Berkeley, Calif.)

4674:

Chandra, K. R. A note on the integer solutions of Mordell's cubic $ax^3+by^3=z^2-k^2$. J. London Math. Soc. **35** (1960), 78–80.

Various cases of the equation in the title are considered and for each it is proved that the equation has an infinity of parametric solutions (x, y, z). Some examples are

$$ax^3+y^3 = z^2-c^{18},$$
$$ax^3+by^3 = z^2-(b^2c^3)^2,$$
$$ax^3+by^3 = z^2-(4abc^3)^2.$$

The latter case, for example, is transformed into a special case of $A(x^3+y^3)+B(z^3-c^3)=0$, which in turn gives rise to $d(x^3+y^3)+z^3=1$. This is a generalization of the case $d=1$ studied recently by Mordell [same J. **30** (1955), 111–113; MR **16**, 798] and by the reviewer [same J. **31** (1956), 275–280; MR **17**, 1187] and is shown to possess an infinite set of parametric solutions.

D. H. Lehmer (Berkeley, Calif.)

4675:

Birch, B. J.; Swinnerton-Dyer, H. P. F. Note on a problem of Chowla. Acta Arith. **5**, 417–423 (1959).

The authors prove the following theorem: For "general" polynomials $f(x)$ we have

$$N(f) = q\left\{\sum_{m=0}^{d}\frac{(-1)^m}{m!}\right\}+O(q^{1/2}),$$

where the constant implied depends only on d. Here $f(x)$ is a polynomial of degree d defined over the finite field k with q elements, and $N(f)$ is the number of distinct values of y in k for which at least one of the roots of $f(x)=y$ is in k.

The problem of estimating $N(f)$ was raised by the reviewer [Bull. Amer. Math. Soc. **58** (1952), 287–303; MR **13**, 915]. *S. Chowla* (Notre Dame, Ind.)

4676:

Malyšev, A. V. A contribution to the theory of ternary quadratic forms. II. On the theorem of Linnik. Vestnik Leningrad. Univ. **14** (1959), no. 13, 63–70. (Russian. English summary)

The principal result in this paper has already been announced in Dokl. Akad. Nauk SSSR **118** (1958), 1078–1080 [MR **21** #33] and is as follows. Let $f(x, y, z)$ be an integral primitive ternary quadratic form with odd coprime invariants $[\Omega, \Delta]$. Let m be an integer prime to $2\Omega\Delta$ for which there is an integral solution of the congruence $f(x, y, z)\equiv m \pmod{8\Omega\Delta}$. Then there exists an s_0, depending only on Ω, Δ, such that if m is divisible by the square of an integer $s\geq s_0$ then m is primitively representable by f. Indeed, the number of primitive representations exceeds $\kappa h(-\Delta m)$, where h is the class-number of primitive binary quadratic forms and κ depends only on Ω, Δ.

The basic idea of the proof is to note that m/s^2 is certainly representable by some form in the same genus as f. The detailed knowledge of the automorphs of the form obtained in the first part [same Vestnik **14** (1959), no. 7, 55–71; MR **21** #2622] allows the representations of m by f to be deduced.

J. W. S. Cassels (Cambridge, England)

4677:

Malyšev, A. V. On the theory of ternary quadratic forms. III. On the representation of large numbers by positive quadratic forms of odd coprime invariants. Vestnik Leningrad. Univ. **15** (1960), no. 1, 70–84. (Russian. English summary)

The principal results of this paper have already been announced in Dokl. Akad. Nauk SSSR **118** (1958), 1078–1080 [MR **21** #33]. The most general result is as follows. Let $f(x, y, z)$ be an integral, primitive, positive definite quadratic form with odd coprime invariants $[\Omega, \Delta]$. Let q be a prime not dividing 2Δ and let g be an integer prime to $2\Omega\Delta$. Let Λ be a conical point-set with vertex at the origin and subtending there an inner solid angle (in the sense of Jordan) $\lambda>0$. Let m be prime to $2\Omega\Delta qg$ and such that $\left(\frac{-\Delta m}{q}\right)=1$. Suppose that there exist integers (x_0, y_0, z_0) such that $f(x_0, y_0, z_0)\equiv m \pmod{8\Omega\Delta g}$. Then there exist constants κ, κ', m_0 depending only on $\Omega, \Delta, q, g, \Lambda$ such that the number t of representations of m by f with $(x, y, z)\in\Lambda$, $(x, y, z)\equiv(x_0, y_0, z_0) \pmod g$ satisfies $\kappa h(-\Delta m)<t<\kappa' h(-\Delta m)$ provided that $m\geq m_0$ (h is the class-number).

The author conjectures that the occurrence of q in these theorems is an accident caused by the method of proof. The method of proof using the generalized theory of quaternions goes back to B. A. Venkov [Izv. Akad. Nauk (6) **16** (1922), 205–246] and Yu. B. Linnik [Izv. Akad. Nauk. Ser. Mat. **4** (1940), 363–402; MR **2**, 348].

J. W. S. Cassels (Cambridge, England)

4678:

Malyšev, A. V. On the theory of ternary quadratic forms. IV. On the connection with the Riemann hypothesis. Vestnik Leningrad. Univ. **15** (1960), no. 7, 14–27. (Russian. English summary)

The principal result of this paper has already been announced in Dokl. Akad. Nauk SSSR **122** (1958), 343–345 [MR **21** #658]. It is that the results of the previous instalment [see preceding review] can be freed of the auxiliary condition $\left(\frac{-\Delta m}{q}\right)=1$ provided that the following weakened form of the generalized Riemann Hypothesis is true: For sufficiently large m there are no zeroes of

$$L(s) = \sum_{1\leq n<\infty}\frac{\chi(n)}{n^s}, \quad \chi(n) = \left(\frac{-4\Omega^2\Delta m}{n}\right)$$

in $|s-1|<(\log\log m)^2\log\log\log m/\sqrt{(\log m)}$. The author deduces from this hypothesis that there exists a prime q with

$$\left(\frac{-\Delta m}{q}\right) = +1, \quad q\leq\kappa\exp\left\{\frac{\mu\sqrt{(\log m)}}{\log\log m}\right\},$$

where κ, μ depend only on Ω, Δ. This is enough for the methods of the preceding instalment to go through with some modifications.

J. W. S. Cassels (Cambridge, England)

4679:

Carlitz, L. Congruences (mod 2^r) for the coefficients of the Jacobi elliptic functions. Math. Z. **72** (1959/60), 307–318.

The Jacobi elliptic function $\mathrm{sn}(x)$ has the expansion

$$\mathrm{sn}(x) = \mathrm{sn}(x, k^2) = \sum_{m=0}^{\infty} A_{2m+1}(k^2)\frac{x^{2m+1}}{(2m+1)!},$$

where $A_{2m+1}(k^2)$ is a polynomial in k^2 with integral coefficients. Let $r \geq 1$ and $2^s \| 2w$. The author proves that $A_n(k^2)$ satisfies

$$\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}(1+k^2)^{(r-s)w}A_{n+2sw}(k^2) \equiv 0 \pmod{2^{sr}}.$$

He also derives congruences of a similar nature for the coefficients of the other Jacobi elliptic functions and some related functions. In two earlier papers [Duke Math. J. **20** (1953), 1–12; Math. Z. **64** (1956), 425–434; MR **14**, 621; **17**, 1057] the author has obtained analogous congruences modulo a power of an odd prime p.

A. L. Whiteman (Princeton, N.J.)

4680:

Rozin, S. M. On null Dirichlet L-series. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 503–508. (Russian)

Soit $p>2$ un nombre premier, $n>0$ entier, $D=p^n$, χ le caractère primitif mod D; $c_i>0$ sont des constantes. Soit $(n_1/(\log n_1)^3)^{1/4} \geq 13$. Pour tout $n>n_1$, on a lorsque $l > p^{2(n\log n)^{3/4}}$: $\sum_{x=N}^{N+l-1}\chi(x) < l\cdot p^{-(n\log n)^{1/4}}$. Application: Pour p constant, $n\to\infty$, $\varepsilon>0$ arbitraire, $L(s,\chi)$ ne s'annule pas dans $|t|<c_1$, $\sigma > 1-c_2/(\log^{3/4+\varepsilon}D\cdot\log\log D)$.

S. Mandelbrojt (Paris)

4681:

Gel'fond, A. O. On the arithmetic equivalent of analyticity of the Dirichlet L-series on the line $\mathrm{Re}\, s=1$. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 145–166. (Russian)

The paper consists of two parts. In the first is given an elementary proof of Dirichlet's theorem which reads

$$\sum_{\substack{n\equiv a(\mathrm{mod}\, k)\\ n\leq x}}\frac{\Lambda(n)}{n} = \varphi^{-1}(k)\ln x + O(1).$$

In addition a generalized Dirichlet theorem is obtained for sequences L obtained from systems of linear substitutions $L_k(x)=mx+qm+k$, $k=0, 1, \cdots, m-1$:

$$\sum_{\substack{p\equiv a(\mathrm{mod}\, k)\\ p\in L,\, p\leq x}}\frac{\ln p}{p} = \frac{1}{\varphi(k)\ln m}\ln\left(1+\frac{m-1}{(r+q)m-r}\right)\ln x + o(\ln x).$$

In the second part it is shown that the absence of zeros of $\zeta(s)$ on the line $\mathrm{Re}\, s=1$ is a consequence of two relations for finite sums:

$$\zeta(1+i\tau)\sum_{n\leq x}\Lambda(n)n^{-(1+i\tau)} = O(\ln^2\tau),$$

$$\zeta(1+i\tau)\sum_{n\leq x}\mu(n)n^{-1}\ln x = \ln x + \sum_{n\leq x}\Lambda(n)n^{-1} + O(\ln^3\tau).$$

From them, with application of some further considerations, comes the estimate (1) $|\zeta(1+i\tau)| > \lambda/\ln^5\tau$, λ being a positive absolute constant. This result is generalized to arbitrary L-series. In addition, by using (1) the bound on the zeros may be moved to the left of the line $\mathrm{Re}\, s=1$, giving the law of distribution of primes with the remainder term $x\exp(-(\ln x)^{1/9})$.

A. I. Vinogradov (RŽMat **1957** #1103)

4682:

Whiteman, Albert Leon. The cyclotomic numbers of order ten. Proc. Sympos. Appl. Math., Vol. 10, pp. 95–111. American Mathematical Society, Providence, R.I., 1960.

Let g be a primitive root of a prime $p=10f+1$. The number (h, k) of solutions of the congruence

$$g^{10x+h}+1 \equiv g^{10y+k} \pmod{p},$$

with $0\leq x, y\leq f$, is called a cyclotomic number of order ten. The problem solved in this paper is to find a formula for the 100 cyclotomic numbers $(0\leq h, k\leq 9)$ as linear combinations of p, x, u, v, w, where

$$16p = x^2+50(u^2+v^2)+125w^2,$$

$$xw = v^2-4uv-u^2, \quad x\equiv 1 \pmod 5. \tag{1}$$

Various equalities exist among the 100 numbers so that it suffices to determine only 22 of them. The problem breaks up into two main cases according to the parity of f. Each main case subdivides 5 ways according to the quintic residue class of 2. Actually only 4 cases are given since the subcases in which 2 is not a quintic residue are easily transformed into each other. For example, in case f is odd and 2 is a quintic residue of p the table gives

$$200(3, 1) = 2p+2+x+75w.$$

Such a formula is not really as explicit as it seems. The number (3, 1) depends on the choice of primitive root, and this dependence is not made explicit. For example, let $p=151$ and $g=114$. The conditions (1) determine $x=-4$ and $|w|=4$ but fail to determine w. Hence, we cannot say whether 20(3, 1) is 600 or 0. Actually, it is 0. The removal of such ambiguities would admittedly be a difficult step forward. Many problems in cyclotomy can tolerate these ambiguities. One of these applications, namely difference sets, is discussed. It is shown that the 10th power residues of a prime cannot form a difference set with or without the inclusion of 0.

D. H. Lehmer (Berkeley, Calif.)

4683:

Lehmer, Emma. On Jacobi functions. Pacific J. Math. **10** (1960), 887–893.

Put

$$R_m = R_m^{(k)}(\alpha) = \sum_{s=1}^{p-2}\alpha^{\mathrm{ind}\, s-(m+1)\mathrm{ind}\,(s+1)},$$

where $\alpha=e^{2\pi i/k}$ and $\mathrm{ind}\, s=\mathrm{ind}_g s$ with respect to some primitive root g of the prime $p=km+1$. Thus R_m depends in general on the choice of the primitive root g and the known explicit results for R_m contain ambiguities of sign. In the present paper explicit results are obtained in the case in which 2 is not a kth power residue of p for $k=3, 4, 5, 6$ and for all m. The case in which 2 is a kth

power residue remains ambiguous. For example, for $k=3$ put

$$\chi(r) = \chi_3(r) = \alpha^{\text{ind } r} \quad (p\nmid r),$$
$$= 0 \quad (p\mid r).$$

Then if $\chi(2)\neq 1$,

$$R_1 = 2B+(B-A)\chi_3(2),$$

where $p=A^2+3B^2$, $A\equiv B\equiv 1 \pmod 3$. If $\chi(2)=1$, $B\equiv 0 \pmod 3$ and the sign of B is not determined; we now have

$$R_1 = (-A\pm B) \pm 2Bw.$$

L. Carlitz (Durham, N.C.)

4684:

Lehmer, D. H. Power character matrices. Pacific J. Math. **10** (1960), 895–907.

In an earlier paper [same J. **6** (1956), 491–499; MR **19**, 7] the writer studied two classes of matrices for which he was able to explicitly determine the characteristic roots, inverse, determinant as well as the general element of any power of the given matrix. The elements of these matrices were simple real functions of the real non-principal character modulo an odd prime. The present paper is concerned with matrices of complex elements based on kth power characters. All matrices considered are square and of order $p-1$, where p is an odd prime.

Let $k>1$, $p=kt+1$ a prime and g a fixed primitive root of p; let $\alpha=\exp(2\pi i/k)$. Put

$$\chi(r) = 0 \quad (p\mid r),$$
$$= \alpha^{\text{ind } h} \quad (p\nmid r).$$

Also put $\varepsilon=\chi(-1)$.

The first type of matrix considered is based on an arbitrary matrix $N=(\alpha_{rs})$ $(r, s=1, \cdots, n)$ with characteristic roots $\rho_1, \cdots, \rho_n$. Put

$$M = (a_{ij}), \quad a_{ij} = \sum_{r,s=1}^{n} \alpha_{rs}\chi(i^{r-1}/j^{s-1}).$$

The general element of M^m and the characteristic roots of M are obtained.

In the second place put $M=(a_{ij})$, $a_{ij}=\chi(i-j)$. Making free use of results from cyclotomy, formulas are found for the general element of M^r $(1\leq r\leq k-1)$ and of M^{rk}. Also the inverse, characteristic polynomial and characteristic roots of M are determined explicitly. To remove certain ambiguities of sign the writer makes use of some recent results of Emma Lehmer [see preceding review].

{It is pointed out by the author that some of the results of the present paper had been anticipated in a paper by the reviewer [Acta Arith. **5** (1959), 292–308; MR **22** #2578].} *L. Carlitz* (Durham, N.C.)

4685:

Breusch, Robert. An elementary proof of the prime number theorem with remainder term. Pacific J. Math. **10** (1960), 487–497.

The author proves the prime number theorem with error term, in an elementary but fairly complicated way. He proves that

$$(1) \qquad \sum_{p^m\leq x} \frac{\log p}{p^m} = \log x + a_x,$$

where $a_x=-A_0+o((\log x)^{-1/6+\varepsilon})$ and A_0 is Euler's constant; as is well known, (1) implies

$$\vartheta(x) = \sum_{p\leq x} \log p = x+o(x(\log x)^{-1/6+\varepsilon}).$$

He uses an asymptotic formula similar to the celebrated formula of A. Selberg.

{For other elementary proofs of the prime number theorem see Selberg [Ann. of Math. (2) **50** (1949), 305–313; MR **10**, 595] and Erdös [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 374–384; MR **10**, 595]; for the prime number theorem with an error term see P. Kuhn [Math. Scand. **3** (1955), 75–89; MR **17**, 587] and Van der Corput [Colloque sur la Théorie des Nombres (Bruxelles, 1955), pp. 163–182, Georges Thone, Liège, 1956; MR **18**, 112].} *P. Erdös* (Budapest)

4686:

Cohn, Harvey. Decomposition into four integral squares in the fields of $2^{1/2}$ and $3^{1/2}$. Amer. J. Math. **82** (1960), 301–322.

Recent experiments by the author [Numer. Math. **1** (1959), 121–134] indicated that every totally positive integer, $\mu=a+b\sqrt{2}$, is the sum of four squares in the field of $\sqrt{2}$ when b is even (a necessary condition). This fact is proved in the present paper by the forthright method of actually giving a formula for the number of representations of such a number as the sum of 4 squares, like that of Jacobi for rational integers and that of Götzky [Math. Ann. **100** (1928), 411–437] for the field of $\sqrt{5}$.

Let G and H denote the sum of the norms of the divisors of μ that have odd norms and even norms respectively. Then the number of representations of μ as the sum of 4 squares is given by $8G$, $32G$, or $48G+6H$ according as the norm of μ is odd, four times an odd number or a multiple of 8.

The method uses Götzky's procedure in that it employs the appropriate theta functions of two complex variables and the fourth power of such series is matched by an Eisenstein series having the same modular transformation group.

For the field of $\sqrt{3}$ the same formulas fail to give the exact number of representations of $a+b\sqrt{3}$ as a sum of 4 squares, there being an additional non-zero error function $E(\mu)$ which, being of lower order than G, permits the conclusion that for all μ of sufficiently large norm 4 squares suffice.

It is conjectured that 3 squares suffice in both fields. *D. H. Lehmer* (Berkeley, Calif.)

4687:

Schinzel, A. On a certain conjecture concerning partitions into sums of three squares. Wiadom. Mat. (2) **1** (1955/56), 205. (Polish)

G. Pall [Amer. Math. Monthly **40** (1933), 10–18] conjectured that every integer of the form $2(8n+1)$, $n\geq 1$, is the sum of three squares of natural numbers. The author disproves this conjecture by pointing out that it is false for $130=2(8\cdot 8+1)$. *H. Halberstam* (London)

4688:

Estermann, T. On the representations of a number as a sum of three squares. Proc. London Math. Soc. (3) **9** (1959), 575–594.

The author achieves a proof of the important formula for the number of representations of n as a sum of 3 squares by the Hardy-Littlewood method, which had

failed hitherto owing to the fact that the double series which occurs in the "singular series" is not absolutely convergent. He overcomes this difficult problem by dealing directly with the double series involved. Previous authors, Maass [Abh. Math. Sem. Univ. Hamburg **12** (1938), 133–162], Streefkerk [Thesis, Free Univ. of Amsterdam, 1943; MR **7**, 414] and P. T. Bateman [Trans. Amer. Math. Soc. **71** (1951), 70–101; MR **13**, 111], also made the evaluation of the representation function by modifications of Hardy's method, based on ideas of Hecke.

S. Chowla (Notre Dame, Ind.)

4689:

Mann, H. B. A refinement of the fundamental theorem on the density of the sum of two sets of integers. Pacific J. Math. **10** (1960), 909–915.

Let $a_0 < a_1 < \cdots$ be a sequence of integers. Denote by $A(n)$ the number of a's not exceeding n. If A and B are such sequences, denote by C the sequence of integers of the form $a_i + b_j$.

The author proves (among others) the following theorem: If $\liminf (A(m) + B(m))/m = 0$, there are infinitely many m's such that

$$C(m) \geq A(m - b_0) + B(m - a_0) - 1.$$

This theorem proves a conjecture of the reviewer in a considerably strengthened form. *P. Erdös* (Budapest)

4690:

Sierpinski, Waclaw. Sur deux suites récurrentes. Matematiche. Catania **12** (1957), 23–30.

Using elementary methods several results concerning the sequence defined by $v_0 = 0$, $v_1 = 1$, $v_{k+2} = -v_{k+1} - 2v_k$, $k \geq 1$, are obtained. These numbers have been considered previously by Dov Jarden and A. Katz [Riveon Lematematica **9** (1955), 72; MR **17**, 585] and in a related form by G. Browkin and A. Schinzel [C. R. Acad. Sci. Paris **242** (1956), 1780–1781; MR **17**, 1055]. The latter proved that $|v_n| > 1$ for $n > 12$. Sample results of the present paper are: (1) $|v_k| + |v_{k+1}| \to \infty$ as $k \to \infty$ (it follows from deeper results of K. Mahler and M. Ward that $|v_k| \to \infty$ as $k \to \infty$); (2) $v_n | v_{nk}$; (3) $v_{n+k} = v_k v_{n+1} - 2v_{k-1} v_n$; (4) $v_k = (-1)^{k+1} 7^{-1/2} 2^{(k+2)/2} \sin k\theta$ where $\theta = \operatorname{Arc\,cos}\, 2^{-3/2}$; and (5) $p | v_{p-1}$ if $p \equiv 1$, 2 or 4 (mod 7) and $p | v_{p+1}$ if $p \equiv 2$, 3 or 6 (mod 7). *L. Moser* (Edmonton, Alta.)

4691:

Chen, Ching-jun. On the representation of a natural number as a sum of terms of the form $x(x+1)\cdots(x+k-1)/k!$. Acta Math. Sinica **9** (1959), 264–270. (Chinese. English summary)

Put $\varphi_k(x) = (-1)^k \binom{-x}{k}$ and denote by $g(\varphi_k)$ the smallest r such that every positive integer is the sum of at most r terms $\varphi_k(x)$ with integral $x \geq 0$. The author improves a previous result by V. I. Nečaev [Izv. Akad. Nauk SSSR. Ser. Mat. **17** (1953), 485–498; MR **15**, 602] and shows that

$$k \log k - k \leq g(\varphi_k) \leq 5(k \log k + 12) \quad (k \geq 12).$$

As the author states, the left-hand inequality was also obtained by L. K. Hua. *K. Mahler* (Manchester)

4692:

Iseki, Shô. On some partition functions. J. Math. Soc. Japan **12** (1960), 81–88.

Let $p_\kappa(n; a, M)$ be the number of partitions of a positive integer n into positive summands of the form $(Ml \pm a)^\kappa$, with integral M, l, a, κ satisfying $M \geq 2$, $0 < a < M$, $(a, M) = 1$, $\kappa \geq 1$. Following essentially the method already used in his paper in Amer. J. Math. **81** (1959), 939–961 [MR **21** #7189], the author establishes the transformation formula for

$$F_\kappa(x; a, M) = \prod_{\substack{r<0 \\ r \equiv \pm a (M)}} (1 - x^{r^\kappa})^{-1} = \sum_{n=0}^{\infty} p_\kappa(n; a, M) x^n.$$

Using a lemma of Hayman [J. Reine Angew. Math. **196** (1956), 67–95; MR **18**, 293] and the reviewer [Trans. Amer. Math. Soc. **89** (1958), 113–128; MR **20** #3840], he then obtains an asymptotic formula for $p_\kappa(n; a, M)$. The particular case $\kappa = 1$ is mentioned separately; but in this case, the author's quoted paper gives actually the exact value, as sum of a convergent series. If n is a prime, the asymptotic formula appears also as a particular case of a formula in the reviewer's quoted paper and the exact formula (if n is a prime) is contained in Livingood's result [Amer. J. Math. **67** (1945), 194–208; MR **6**, 259].

E. Grosswald (Philadelphia, Pa.)

4693:

Debreu, Gerard. On "An identity in arithmetic". Proc. Amer. Math. Soc. **11** (1960), 220–221.

In an article in Bull. Amer. Math. Soc. **65** (1959), 123–124 [MR **21** #3910], H. D. Block and J. Marschak obtained an arithmetical identity, which was intuitively clear from certain probability considerations. However, they were unable to find the required theorem in measure theory which would convert the intuitive considerations into a rigorous proof of the identity. Consequently they provided a proof by induction. This note supplies a purely probabilistic proof of the identity.

Benjamin Epstein (Palo Alto, Calif.)

4694:

Cassels, J. W. S. On a problem of Steinhaus about normal numbers. Colloq. Math. **7** (1959), 95–101.

The real number ξ is called normal with respect to the integer b ($b > 1$) as base, if in the "decimal" expansion of ξ to the base b every digit occurs with the same asymptotic frequency. H. Steinhaus [*The new Scottish book*, Wroclaw 1946–1958, Problem 144, p. 14] asked whether normality with respect to infinitely many b's implies normality with respect to all other b's. The author answers this question in the negative, constructing continuously many ξ's which are non-normal with respect to 3, but normal with respect to every integer which is not a power of 3. Actually he proves the following: Let $f(x)$ be defined, for $0 \leq x < 1$, by $f(\sum_1^\infty \varepsilon_j 2^{-j}) = \sum_1^\infty \varepsilon_j 3^{-j}$ for all sequences ε_j which only take the values 0 or 1 (and $\varepsilon_j = 0$ infinitely often). Then for almost all x the number $f(x)$ is normal with respect to every integer which is not a power of 3. And $f(x)$ is obviously not normal with respect to 3.

N. G. de Bruijn (Eindhoven)

4695:

Kesten, Harry. Uniform distribution mod 1. Ann. of Math. (2) **71** (1960), 445–471.

Let $f(x)$ be the characteristic function of the interval

$0<a\leqq x\leqq b<1$, and $f(x+1)=f(x)$. The problem of studying the asymptotic distribution of the sum $\sum_{k=1}^{N}f(y+kx)$, when x and y are independent random variables uniformly distributed over [0, 1], was suggested by the reviewer [Research Problem No. 6, Bull. Amer. Math. Soc. **64** (1958), 60]. The author shows that for rational $b-a$ the quantity $(\log N)^{-1}(\sum_{k=1}^{N}f(y+kx)-N(b-a))$ has a Cauchy distribution asymptotically. The proof requires some detailed analysis. *R. D. Bellman* (Santa Monica, Calif.)

4696:

Pyateckiĭ-Šapiro, I. I. On the distribution of the fractional parts of the exponential function. Moskov. Gos. Ped. Inst. Uč. Zap. **108** (1957), 317–322. (Russian)

The results of the author [Izv. Akad. Nauk SSSR Ser. Mat. **15** (1951), 47–52; MR **13**, 213] and A. G. Postnikov [Dokl. Akad. Nauk SSSR **86** (1952), 473–476; MR **14**, 359] are strengthened. Let α be a real number, $q>1$ be an integer, $P_\nu(\alpha,\Delta)$ be the number of fractional parts $\{\alpha q^k\}$ $(k=0,1,\cdots,\nu=1)$ belonging to the interval $\Delta\subset(0,1)$, $|\Delta|$ be the length of the interval Δ, and $\varphi(t)$ be a monotone function defined on the interval [0, 1]. If

$$\liminf_{t\to 0}\varphi(t)t^{\epsilon-1}=0$$

for arbitrary $\epsilon>0$, and if there exists a constant C such that for any interval Δ

$$\limsup_{\nu\to\infty}P_\nu(\alpha,\Delta)\leqq C\varphi(|\Delta|), \tag{1}$$

then the sequence $\{\alpha q^k\}$ is uniformly distributed. If however $\liminf_{t\to 0}\varphi(t)t^{\epsilon-1}>0$ for some $\epsilon>0$, one may always take α such that (1) is fulfilled but the fractional parts $\{\alpha q^k\}$ are non-uniformly distributed.

I. P. Kubilyus (RŽMat **1959** #2307)

4697:

Aigner, Alexander. Einige handliche Regeln für biquadratische Reste. Math. Nachr. **17**, 219–223 (1959).

The symbol $(q/p)_4$ may be evaluated by biquadratic reciprocity as $([-1]^{\frac{1}{2}p}+2yp^{1/2}/q)_2$ for primes p, q, where $(q/p)_2=1$, $p\equiv 1 \pmod 4$, and $p=x^2+4y^2$. From this rule, $(-1)^{(q-1)/2}q$ is a biquadratic residue of p if $q|y$, or, if $q|x$, exactly when $q\equiv\pm 1 \pmod 8$. The rule is exhaustive only when $q=3, 5, 7$. See Hasse's *Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper*, Teil II [Teubner, Berlin, 1930; p. 69].

The author extends such rules by elementary techniques, using composite values of $q=6, 10, 15, 14, 21, 35$, where the factors of q, unlike q, are not individually quadratic residues of p. For example, for $q=15$, $p\equiv 17 \pmod{60}$, then $(-15/p)_4=1$ exactly when $15|x\pm 2y$. He also shows that no such easy rule will work for $q=39$. The reduction to quadratic symbols is accomplished without biquadratic reciprocity by a trick of the following type (in the above example): $p=5a^2+12b^2$; hence $(-15)^{1/2}\equiv 5a/2b \pmod p$.

H. Cohn (Tucson, Ariz.)

4698:

Habicht, Walter. Ein elementarer Beweis des kubischen Reziprozitätsgesetzes. Math. Ann. **139**, 343–365 (1960).

The proof of the cubic reciprocity law, which the author gives here, is analogous to the most elementary of Gauss's seven proofs of the quadratic reciprocity law.

T. Estermann (London)

4699:

Frölich, A. The genus field and genus group in finite number fields. I, II. Mathematika **6** (1959), 40–46, 142–146.

Let K be a finite algebraic number field. The author defines the genus field $\bar{K}$ of K to be the maximal unramified extension of K of the form $K\Omega$, where Ω is an absolutely abelian field. $\bar{K}$ can also be described as the largest unramified extension of K whose norm group (in the ideal-theoretic form of class field theory) can be characterized by rational congruences. The genus group of K is the factor group of K-ideals modulo the group of ideals to which $\bar{K}$ is class field: its order divides the class number of K. Using earlier results of his [Mathematika **4** (1957), 113–121; MR **20** #4538], he derives a formula describing the norm group of $\bar{K}$ in terms of the local behavior at the primes dividing the discriminant of K. He applies these results to several interesting examples such as these: In theorems 6 and 7 of I, he computes the genus groups of normal non-abelian extensions of degree 6 over the rational field, and of the splitting field of x^n-m over the rational field. Theorem 8: For arbitrary positive integers n and q there exists a normal number field K whose Galois group is the symmetric group on n symbols and whose class number has the factor n^q.

In the second paper he studies relations between the genus groups of a field and of its subfield and applies these results to determining genus groups of fields $P(m^{1/l^n})$ where $P=$ rational field, m an l-power-free integer.

G. Whaples (Bloomington, Ind.)

4700:

Barrucand, Pierre. Sommes de Gauss et séries singulières de Hardy pour les cubes. C. R. Acad. Sci. Paris **250** (1960), 4249–4251.

The Gauss sum in the title is defined by $S_{h,q}=\sum_{a \bmod q}\exp(2\pi iha^3/q)$. This paper is concerned with the problem of evaluating the singular series $\mathfrak{S}_\nu(n)=\sum_{q=1}^{\infty}B(\nu,q,n)/q^\nu$, where $B(\nu,q,n)=\sum'_{h \bmod q}S_{h,q}^\nu\times\exp(-2\pi inh/q)$ and h runs over a reduced residue system modulo q. The principal formula obtained may be stated as follows. Let p be a prime $\equiv 1 \pmod 3$ so that $4p=L^2+27M^2$ with $M>0$ and $L\equiv 1 \pmod 3$; let z_a, z_b, z_c denote the three roots of the period equation $z^3-3pz-pL=0$ and put $z_a^\nu+z_b^\nu+z_c^\nu=3\phi_\nu$; let $\varepsilon(n)=\mathrm{Re}\,(n|p)_3$ and $\varepsilon'(n)=3^{-1/2}\,\mathrm{Im}\,(n|p)_3$, where $(n|p)_3$ represents the cubic residue character of n with respect to p. (1) If $p\nmid n$, then $B(\nu,p,n)=-\phi_\nu+\varepsilon(n)\phi_{\nu+1}+\varepsilon'(n)3^{-1}M^{-1}(4p\phi_\nu+L\phi_{\nu+1}-2\phi_{\nu+2})$. (2) If $p|n$, then $B(\nu,p,n)=(p-1)\phi_\nu$.

A. L. Whiteman (Princeton, N.J.)

4701:

Jarník, Vojtěch. Eine Bemerkung über diophantische Approximationen. Math. Z. **72** (1959/60), 187–191.

Let $\Theta=(\theta_{ij})$ $(1\leqq i\leqq n,\ 1\leqq j\leqq m)$ be a matrix of real numbers. For $t\geqq 1$ write

$$\Psi_\Theta(t)=\min_{0<x\leqq t}\Big(\max_{1\leqq i\leqq n}\Big|\sum_{1\leqq j\leqq m}\theta_{ij}x_j+x_{m+i}\Big|\Big),$$

where $x=\max(|x_1|,\cdots,|x_m|)$ and the $x_1,\cdots,x_{m+n}$ are integers. Let $\phi(t)$ be a positive decreasing function of t, decreasing however fast, and let $\phi_1(t)$ be a positive function of t which tends to infinity, however slowly, when t tends to infinity. The author shows that there exist algebraically independent sets of θ_{ij} such that

$$(*)\qquad \Psi_\theta(t) = O(\phi_1(t)/t) \quad (m = 1, n > 1),$$
$$\Psi_\theta(t) = O(\phi(t)) \qquad (m > 1).$$

Indeed the projection on any coordinate axis of the set of such algebraically independent θ_{ij} in any given open set has the power of the continuum.

Results of this kind but with linear independence instead of algebraic independence were obtained by Hinčin [Rend. Circ. Mat. Palermo **50** (1926), 170–195]. For $n<m$ the existence of algebraically independent θ_{ij} with (*) was shown by Chabauty and Lutz [C. R. Acad. Sci. Paris **231** (1950), 887–888; MR **12**, 483] but their results were weaker for $m \leqq n$. Since the author [Trav. Inst. Math. Tbilissi **3** (1938), 193–212] has shown in the case $m=1$, $n>1$ that $\limsup t\Psi_\theta(t) = +\infty$ when and only when at least two of the numbers $\theta_{11}, \cdots, \theta_{n1}$, are linearly independent, it follows that the author's results are the best possible. *J. W. S. Cassels* (Cambridge, England)

4702:

Walfisz, Arnold. Über Gitterpunkte in vierdimensionalen Ellipsoiden. Math. Z. **72** (1959/60), 259–278.

With the help of estimates of exponential sums due to Winogradoff the author sharpens the known estimates for the error term in the expression for the number of lattice points in the 4-dimensional sphere $x_1^2+x_2^2+x_3^2+x_4^2 \leqq x$ to $O(x \log^{2/3} x)$. More generally, he achieves this for a quaternary definite form $Q(x_1, x_2, x_3, x_4)$ with rational coefficients. He has not used recent work of Koroboff which at first seemed necessary to obtain his sharp estimate. *S. Chowla* (Notre Dame, Ind.)

4703:

Mulholland, H. P. On the product of n complex homogeneous linear forms. J. London Math. Soc. **35** (1960), 241–250.

Let $x_1, x_2, \cdots, x_n$ be n homogeneous linear forms in n variables with associated matrix A, and let $L(A) = \inf (x_1 x_2 \cdots x_n)^{1/n}$, for integral values of the variables not all zero. Let $b_\gamma(n)$ be the class of $n \times n$ matrices A, with $|\det A| = 1$, having γn rows complex (in pairs of conjugates) and the remaining $(1-\gamma)n$ rows real. Let $M_\gamma(n)$ be the supremum of $L(A)$ for $A \in b_\gamma(n)$. Also put $\mathscr{M}_c = \limsup M_\gamma(n)$ for $n \to \infty$ and $\gamma \to c$, where $0 \leqq c \leqq 1$. C. A. Rogers [Acta Math. **82** (1950), 185–208; MR **11**, 501] showed that $\mathscr{M}_0 \leqq \pi/4e^{3/2} = 1/5.70626\cdots$. The author proves that, for $0 \leqq c \leqq 1$,

$$\mathscr{M}_c \leqq (\tfrac{1}{2}\sqrt{\pi})^{2-2c} e^{-3/2} \leqq 2\pi^{-1/2} e^{-3/2} = 1/3.97179\cdots.$$

This improves on a result of Blichfeldt which yields $\mathscr{M}_c \leqq (\pi e)^{-1/2} = 1/2.92221\cdots$. The author also indicates how his result can be improved to give $\mathscr{M}_c \leqq 1/3.985900\cdots$. The proof depends upon estimations of the number

$$C_m(k) = \sup \{ \prod_{\rho \neq \sigma} |z_\rho - z_\sigma|^{1/(m^2-m)} / (\sum |z_\rho|^k/m)^{1/k} \},$$

where $k>0$ and $z_1, z_2, \cdots, z_m$ are complex variables not all zero. By using the theory of the logarithmic potential and an inequality for integrals, analogous to one established for real variables by Rogers, and proved earlier by the author [same J. **33** (1958), 260–270; MR **21** #722], it is shown that $C_m(k)$ decreases as m increases and that

$$(2e^{-1/2})^{1/k} m^{(1-\eta)/2m} \leqq C_m(k) \leqq (2e^{-1/2})^{m/(km-k)} \left(\frac{m^{m(m-1)}}{m-1}\right)^{1/\kappa},$$

where $\kappa = \min(k, 1)$ and $\eta = \eta(m, k) \to 0$ as $m \to \infty$. In particular, $C_m(k) \to (2e^{-1/2})^{1/k}$ as $m \to \infty$. The first estimate for $\mathscr{M}_c$ is obtained by taking $k=1$, and the second by taking $k=5/6$. *R. A. Rankin* (Glasgow)

FIELDS

See also 4639, 4744.

4704:

Popovič, Konstantin P. [Popovici, Constantin P.]. Integral polynomials irreducible modulo p. Rev. Math. Pures Appl. **4** (1959), 369–379. (Russian)

The author gives an algorithm for constructing all the irreducible polynomials in one variable over the prime field of characteristic p. *E. Lluis* (Mexico City)

4705:

Mahler, K. On a theorem by E. Bombieri. Nederl. Akad. Wetensch. Proc. Ser. A **63** = Indag. Math. **22** (1960), 245–253.

Here α and β denote roots of real or complex polynomials f and g of heights $H(f)$ and $H(g)$, degrees m and n, and non-zero resultant $R(f, g)$. Then

$$|\alpha - \beta| \geqq |R(f,g)|/|H(f)^n H(g)^m C_{n,m}.$$

If α and β are distinct roots of the same polynomial f, and if $D(f)$ is the non-zero root-discriminant, then

$$|\alpha - \beta| \geqq |D(f)|/|H(f)^{2m-1} C_m.$$

The constants C are not complicated, e.g., $C_m/(m+1)^{4m} 4^{m^2} = 1$ if f has only real zeros and $= (m+1)^{m^2}$ otherwise. This improves results of A. Brauer [Jber. Deutsch. Math. Verein. **38** (1929), 47] and Bombieri [Boll. Un. Mat. Ital. (3) **13** (1958), 351–354; MR **21** #42] in regard to strength of inequality and extension to non-algebraic numbers, by use of the resultant and discriminant.

H. Cohn (Tucson, Ariz.)

4706:

Grebenyuk, D. G. On the theory of algebraic integers depending on a root of an irreducible equation of fourth degree. Izv. Akad. Nauk UzSSR. Ser. Fiz.-Mat. **1958**, no. 6, 27–47. (Russian. Uzbek summary)

The author [see Byull. Sr.-Az. Gos. Univ. **11** (1925), 13–43] showed that the bi-quadratic field generated by a root of $f(\theta) = 0$ has a basis of the form $\omega_1 = 1$, $\omega_2 = (\theta - \tau)/\delta$, $\omega_3 = (\theta^2 + 2\tau\theta - 3\tau^2 - r)/\delta^2\sigma$ and $\omega_4 = [\theta^3 + \tau\theta^2 + (\tau^2 - r)\theta + \tau^3 - r\tau - s]/\delta^3\sigma^2\kappa$, where τ, δ, σ, κ satisfy $\delta^4\sigma^3\kappa^2 | f(\tau) = \tau^4 - r\tau^2 - s\tau - t$, $\delta^3\sigma^2\kappa | f'(\tau)$, $\delta^2\sigma | [f''(\tau)/2]$, $\delta | [f'''(\tau)/6]$, and δ, σ, and τ are chosen as maximal divisors of the root-discriminant of θ. In this work the author goes through the actual determination of these auxiliary quantities. *H. Cohn* (Tucson, Ariz.)

4707:

Frölich, Albrecht. Discriminants of algebraic number fields. Math. Z. **74** (1960), 18–28.

Let K be a number field and Λ a number field containing K. The relative discriminant of the extension Λ|K is then classically defined as a certain ideal; if K happens to be the field of rational numbers, however, the relative discriminant is defined as a certain number in K. The author introduces a new kind of discriminant which unifies these classical definitions. He does this by considering the group of idèles in K, and defines the discriminant $\mathfrak{d}(\Lambda, K)$ as a certain class of rational idèles modulo the squares of the unit-idèles. The precise formulation is given in terms of the local discriminants arising from the various prime divisors in K. The classical discriminant is then the ideal derived from $\mathfrak{d}(\Lambda, K)$. The usefulness of this new concept is shown by the fact that the ring of algebraic integers in Λ has an integral basis over the ring of algebraic integers in K if, and only if, $\mathfrak{d}(\Lambda, K)$ is principal. Among other things he discusses the discriminant of quadratic extensions of K and finds that they have properties that are natural generalizations of the well known ones for the case where K is the rational field. As another illustration he obtains, in complete analogy to the rational case, the classification of cubic fields over K into cyclic and non-cyclic ones.

H. W. Brinkmann (Swarthmore, Pa.)

4708:

Frölich, Albrecht. Ideals in an extension field as modules over the algebraic integers in a finite number field. Math. Z. **74** (1960), 29–38.

This paper is a continuation of the one reviewed in the preceding review. If $\mathfrak{A}$ is an ideal in the extension field Λ the author introduces what he calls the discriminant $\mathfrak{d}(\mathfrak{A})$ of $\mathfrak{A}$ in K; if $\mathfrak{A}$ is the ring of all integers in Λ this becomes the discriminant $\mathfrak{d}(\Lambda, K)$ defined in the first paper. He also introduces what he calls a module invariant $C(\mathfrak{A})$, which is a certain ideal class in K, and shows that this can be determined in terms of $\mathfrak{d}(\mathfrak{A})$ and the relative norm of $\mathfrak{A}$. One interesting result is that he is able to obtain a neat characterization of the absolute class field of K in terms of these concepts.

H. W. Brinkmann (Swarthmore, Pa.)

4709:

Endler, Otto. Kennzeichnung abelscher Körpererweiterungen vom Grad p^m und von vorgegebenem Erweiterungstypus über einem Teilkörper des Grundkörpers. II. Kennzeichnung im Restklassenkörper. Math. Nachr. **20** (1959), 127–151.

Let k be a field of characteristic p which is Galois over a subfield k_0. In the former paper I [see Math. Nachr. **17** (1958), 73–92; MR **20** #3854] abelian extensions K of k of type $(p^{m_1}, \cdots, p^{m_r})$ which are Galois over k_0, with a given extension type, have been characterized by a certain type of r-tuples of Witt vectors over k, in application of the Artin-Schreier-Witt theory. The present part II deals with the case where k is complete for a discrete valuation, and gives a characterization of K by a certain family of infinite, but essentially finite, series in the residue field κ (assumed to be perfect) of k. The procedure consists, roughly speaking, in expressing Witt vectors in the above general characterization by power series, in a variable t, over the canonical representative field of κ, and then in combining some arithmetical consideration on Witt vectors with the result in I, to pass to a partial matrix of the matrix formed by the coefficients in the power series. Also the effectiveness of the characterization is discussed.

T. Nakayama (Nagoya)

4710:

Endler, Otto. On the inverse problem of Galois theory. An. Acad. Brasil. Ci. **31** (1959), 331–332. (Portuguese)

Given an infinite field K and a finite group Γ, choose G^* isomorphic to Γ, where G^* is a group of K-automorphisms of the field $N^* = K(t_1, \cdots, t_k)$; here, $t_1, \cdots, t_k$ are algebraically independent over K for a suitable $k \geq 1$. A fundamental polynomial for G^* is defined as

$$F_b^* = \prod [X - \sigma^*(\sum t_i b_i)],$$

where the product extends over $\sigma^* \in G^*$, where the sum extends over the range $1 \leq i \leq k$, and where $b_1, \cdots, b_k \in K$ are such that F_b^* is separable. The author gives, without specific references, a very abbreviated sketch of the question of determining the Galois extensions N/K of an infinite field K, with Galois group G isomorphic to Γ. He describes his sketch as being "based on recent work of W. Krull, which generalizes classical results of D. Hilbert, E. Noether, and F. Klein", and discusses the problem by means of "fundamental polynomials".

R. G. Stanton (Waterloo, Ont.)

4711:

O'Keefe, Kathleen B. A property of the differential ideal y^p. Trans. Amer. Math. Soc. **94** (1960), 483–497.

Let p be a positive integer, let y be a differential indeterminate, let $[y^p]$ be the differential ideal generated by y^p in the ring of differential polynomials in y having rational numbers as coefficients. It is shown that the smallest integer q for which $y_i^q \equiv 0\ [y^p]$ (where subscripts denote differentiation) is $2p-1$ if $i=1$, and $3p-2$ if $i=2$. The second result is new. It is conjectured that for arbitrary i the answer is $(i+1)(p-1)+1$.

H. Levi (New York)

ABSTRACT ALGEBRAIC GEOMETRY

4712:

Galbură, G. [Galbură, Gh.]. The ring of rational equivalences of an algebraic variety. Rev. Math. Pures Appl. **4** (1959), 351–355. (Russian)

Let W be a subvariety of a projective variety V (Weil's terminology) and let $\mathfrak{A}_W$, $\mathfrak{A}_V$ be the respective rings of rational equivalence classes. The author analyses the homomorphism $\iota_V{}^W$ of the additive group of $\mathfrak{A}_W$ into the additive group of $\mathfrak{A}_V$ induced by inclusion and the ring homomorphism $\sigma_W{}^V : \mathfrak{A}_V \to \mathfrak{A}_W$ induced by intersection with W. Certain relations between them are obtained, the most elaborate of them being

$$\iota_Z{}^V(\sigma_Z{}^{W_1}a_1 \cdot \sigma_Z{}^{W_2}a_2)_Z = (\iota_V{}^{W_1}a_1 \cdot \iota_V{}^{W_2}a_2)_V,$$

$$\iota_V{}^{W_1}(a_1 \cdot \iota_{W_1}{}^Z(c \cdot \sigma_Z{}^{W_2}a_2)_Z)_{W_1} = \iota_V{}^{W_2}(a_2 \cdot \iota_{W_2}{}^Z(c \cdot \sigma_Z{}^{W_1}a_1)_Z)_{W_2},$$

$$\iota_Z{}^V(\sigma_Z{}^{W_1}a_1 \cdot \sigma_Z{}^{W_1}b_1, \sigma_Z{}^{W_2}a_2 \cdot \sigma_Z{}^{W_2}b_2)_Z = (\iota_V{}^{W_1}(a_1 \cdot b_1) \cdot \iota_V{}^{W_2}(a_0 \cdot b_2))_V,$$

where W_1, W_2 are simple subvarieties of V whose points are simple on V, and $Z = W_1 \cap W_2$ is proper and irreducible.

E. Lluis (Mexico City)

4713:

Galbură, Gh. Sur l'anneau d'équivalence rationnelle d'une variété algébrique. Acad. R. P. Romîne. Stud. Cerc. Mat. **10** (1959), 325–330. (Romanian. Russian and French summaries)

Romanian version of #4712. *E. Lluis* (Mexico City)

4714:

Gutwirth, A.; Jabotinsky, E. An alignment matrix and its applications to plane cubic curves and their rational points. Bull. Res. Council Israel. Sect. A **6** (1956), 11–41.

An alignment matrix is defined to be a matrix of three rows (x_i, y_i, z_i) $(i=1, 2, 3)$ such that the sum $(=\mu)$ and the product $(=\lambda)$ of the elements in each row is the same, the coordinates being taken from an arbitrary field K. The authors enunciate a large number of identities which hold amongst the elements of an alignment matrix, perhaps the most striking being that

$$(x_1x_2x_3)^{1/3}+(y_1y_2y_3)^{1/3}+(z_1z_2z_3)^{1/3}=0,$$

for an appropriate choice of the cube roots, provided only that two of the cube roots are in K. These identities are proved on the assumption that the alignment matrix is non-degenerate, in a sense which is defined; but the authors also investigate how far they continue to be valid for degenerate alignment matrices. As they point out, the three rows of a non-degenerate alignment matrix are the homogeneous coordinates of the intersection of a line with the cubic $\mu^3xyz=\lambda(x+y+z)^3$ of genus 1, and this places their results in a wider setting. The authors give a geometric interpretation of degenerate alignment matrices. Finally, they discuss when a general curve of genus 1 can be brought into their canonical form, and the condition under which the group of the rational points (over K) on the curve contains elements of order 3 and 9 [cf., e.g., Selmer, Math. Scand. **2** (1954), 227–236; MR **16**, 740].

J. W. S. Cassels (Cambridge, England)

4715:

Arima, Satoshi. Certain generators of non-hyperelliptic fields of algebraic functions of genus ≥ 3. Proc. Japan Acad. **36** (1960), 6–9.

Suppose that K is a field of algebraic functions of one variable over the algebraically closed field Ω of characteristic 0. Denote by E_i, $1\leq i\leq 2^{2G}$, divisors of K which represent the $2^{2G}=m$ divisor classes of order 2 in K, G being the genus of K. Set $E_i^2=(x_i)$ and define $k=\Omega(x_1, \cdots, x_m)$, $[K:k]=n$. The author proves that $K=k$ if K is not hyperelliptic and $G\geq 3$. Furthermore, if K is hyperelliptic and $G\geq 3$, then $K=k$ or $[K:k]=2$ with rational k; if $G=2$, then n may equal 1, 2 or 4, and in the latter two cases k must be rational. The proofs depend on the inequality $n\leq 2+(G-3/2)^{-1}$ if $n>1$ and $G\geq 2$. The inequality is obtained as a consequence of the Riemann-Hurwitz formula for G in terms of the genus of k and the local ramification degrees of K/k; it is achieved by careful examination of the elements x_i considered as divisors in K and k, respectively. *O. F. G. Schilling* (Chicago, Ill.)

4716:

Mallol, Rafael. Über die Existenz eines singularitätenfreien Modells einer algebraischen Kurve. Math. Ann. **140** (1960), 344–350.

Another proof, using explicit computations with Dedekind-Weber normal bases, of the well-known fact that a regular extension field of transcendence degree one of a given base field Δ does not drop in genus under any extension of Δ if and only if the field has an absolutely nonsingular projective model over Δ. For a generalization see Theorem 11 of the reviewer's paper in Ann. of Math. (2) **56** (1952), 169–191 [MR **14**, 80].

M. Rosenlicht (Berkeley, Calif.)

4717:

Mattuck, Arthur. Reduction mod p of p-adic divisor classes. J. Reine Angew. Math. **200** (1958), 45–51.

Es sei K ein Funktionenkörper einer Variablen über einem p-adischen Zahlkörper k. Vorausgesetzt wird, daß K/k eine nichtausgeartete (="reguläre") Spezialisierung $K\to K'$ über p besitzt, im Sinne der Deuringschen Reduktionstheorie. [Vgl. Deuring, Math. Z. **47** (1942), 643–654; MR **7**, 362.] Es wird gezeigt, daß jeder separable Primdivisor $\mathfrak{p}'$ von K' als Bild eines Primdivisors $\mathfrak{p}$ von K auftritt. Als Folgerung ergibt sich, daß die Divisorklassengruppe 0-ten Grades $\mathfrak{D}$ von K bei der Spezialisierung $K\to K'$ auf die entsprechende Gruppe $\mathfrak{D}'$ von K' abgebildet wird.—$\mathfrak{D}$ läßt eine direkte Zerlegung $\mathfrak{D}=\mathfrak{D}_0\oplus\mathfrak{E}$ zu, wobei $\mathfrak{D}_0$ die Gruppe der Elemente von endlicher und zu p primer Ordnung aus $\mathfrak{D}$ bezeichnet (p ist die zu p gehörige Primzahl), und wo $\mathfrak{E}$ eine endliche p-Erweiterung einer Gruppe $\mathfrak{Z}$ ist, die ihrerseits isomorph ist zu einem Vektorraum über den ganzen p-adischen Zahlen; man nennt $\mathfrak{Z}$ eine "Lutzsche Untergruppe" von $\mathfrak{D}$. Es wird gezeigt, daß $\mathfrak{D}_0$ bei der Spezialisierung $K\to K'$ auf die entsprechende Gruppe $\mathfrak{D}_0'$ abgebildet wird, während $\mathfrak{E}$ auf die Gruppe $\mathfrak{D}_p'$ der Elemente mit durch p teilbarer Ordnung aus $\mathfrak{D}'$ abgebildet wird.—Diese Resultate werden im elliptischen Spezialfall diskutiert. Aus Resultaten von Lutz [dasselbe J. **177** (1937), 238–247] läßt sich ablesen, daß $\mathfrak{Z}$ der genaue Kern in $\mathfrak{D}$ der Spezialisierungsabbildung ist; hieraus ergeben sich gewisse Strukturaussagen über $\mathfrak{E}$, und insbesondere über die Gruppe $\mathfrak{D}_p$ der Elemente mit durch p teilbarer endlicher Ordnung aus $\mathfrak{D}$.

P. Roquette (Tübingen)

4718:

Barsotti, Iacopo. Moduli canonici e gruppi analitici commutativi. Ann. Scuola Norm. Sup. Pisa (3) **13** (1959), 303–372.

In this paper, the author gives a complete exposition of the theory of commutative formal Lie (=analytic) groups, which is equivalent to the theory of commutative hyperalgebras; and the latter is itself shown to be equivalent to the theory of a special type of modules.

In § 1, he starts with the description of these modules: let K' be a complete field of characteristic 0, for a discrete valuation v with algebraically closed residual field k of characteristic $p>0$; v is supposed to be normalized, and it is also supposed that there exists $\omega\in K'$ such that $v(\omega)=1$ and $\omega^e=p$; K is the ring of integers of K' and $a\to a^\varkappa$ the Frobenius automorphism of K such that $\omega^\varkappa=\omega$ and $\mu(a^\varkappa)=(\mu a)^p$, where $\mu: K\to k$ is the natural mapping. Let T be the ring of formal power series in one indeterminate t, with coefficients in K, and with multiplication defined by the rule $at=ta^\varkappa$ for $a\in K$; let T^* be the similar ring, where now a finite number of powers t^k with $k<0$ may appear. It is shown how one can give a

complete description of finitely generated torsion left T^*-modules; such a module is a direct sum of a finite number of indecomposable ones; the latter are isomorphic either to a $T^*/T^*\omega^s$, or to a $T^*/T^*(\omega^r - t^s)$, where $(r, s) = 1$; the second type of modules consists of simple modules, whose sfield of endomorphisms is a cyclic algebra over the extension $Q_p(\zeta_s, \omega)$ of the p-adic field, where ζ_s is a $(p^s - 1)$th primitive root of unity.

In § 2, the author deals with those finitely generated T-modules M which he calls canonical: M must be a submodule of a T^*-module, and satisfy $\omega M \subset tM$. For such a module one defines its dimension as the minimum number of generators; M is said to be equidimensional if M and ωM have the same dimension; and then the length of $M/\pi M$ (with $\pi = t^{-1}\omega$), which is finite, is called the codimension of M. Various classes of canonical modules are introduced, according to the behavior of the submodules $\pi^k M$.

In § 3 the author introduces a duality between canonical T-modules, which cannot be described here more explicitly for lack of space.

In § 4 he introduces the notion of commutative hyperalgebra over k, and the dual notion (called "hyperdomain" by the author) which is also dual to that of commutative formal Lie group.

The object of § 5 is the description of the fundamental one–one correspondence between commutative hyperalgebras over k and canonical T-modules (with $e = 1$); K is then isomorphic to the ring $W(k)$ of Witt-vectors over k, and T acts naturally on any ring $W(A)$ of Witt-vectors over a commutative algebra A over k. A canonical sub-T-module M of $W(A)$ is said to be freely enveloped by A if there is a minimal system of generators $x_t = (x_{tj})$ of M such that the x_{tj} constitute a p-basis for A over k. It is shown that any canonical T-module is isomorphic to such a T-module of Witt-vectors. Furthermore, if $M \subset W(A)$ is freely enveloped by A, there is a unique diagonal mapping $\Delta: A \to A \otimes A$ which turns A into a hyperalgebra, and for which M is exactly the set of Witt-vectors x such that $\Delta x = x \otimes 1 + 1 \otimes x$ (Δ being naturally extended to a homomorphism $W(A) \to W(A) \otimes W(A)$); and conversely for any hyperalgebra A the set M of these vectors x is a canonical T-module. In a similar way one links homomorphisms of hyperalgebras to homomorphisms of canonical T-modules.

Finally, in §§ 6 and 7, the author develops several algorithms on Witt-vectors, with which he can interpret the duality between canonical T-modules defined in § 3 in terms of the duality between hyperalgebras and "hyperdomains". *J. Dieudonné* (Paris)

4719:

Kuhlmann, Norbert. Zur Theorie der Modifikationen algebraischer Varietäten (birationale Transformationen). Schr. Math. Inst. Univ. Münster No. 14 (1959), iii + 49 pp.

In this paper the author gives a general construction for "blowing up" a coherent sheaf of ideals on an algebraic variety. This generalizes the classical monoidal transformation. The analogous problem for complex spaces is discussed with particular reference to the complex structure of algebraic varieties over the complex field.

M. F. Atiyah (Cambridge, England)

LINEAR ALGEBRA

See also B5118.

4720:

Easterfield, T. E. The characteristic roots of a matrix: a correction. Duke Math. J. **23** (1956), 635–637.

The paper corrected is by Sen-Ming Leng [same J. **19** (1952), 139–154; MR **14**, 7]. Some of Leng's proofs are asserted to be invalid, and counterexamples offered for the results.

4721:

Parodi, Maurice. Sur la formation de matrices stochastiques ayant pour valeurs caractéristiques celles d'une matrice donnée. C. R. Acad. Sci. Paris **250** (1960), 3258–3259.

Given an nth order (square) matrix A with real entries and characteristic values of modulus at most 1, the author gives a construction which, in some cases, yields an $(n+1)$th order stochastic matrix (non-negative entries and row sums equal to 1), whose characteristic values are those of A and the number 1. If applied to the matrix $A = (a_{jk})$ with row sums r_j, the method is successful if and only if one can find n numbers α_j with sum α such that $\alpha_j \geq 0$, $1 - \alpha \geq 0$, $a_{kj} + \alpha_j \geq 0$ and $1 - r_j - \alpha \geq 0$ for $j, k = 1, 2, \cdots, n$.

R. Steinberg (Los Angeles, Calif.)

4722:

Slepian, Paul. On paramount matrices. Quart. Appl. Math. **18** (1960/61), 263–269.

A paramount matrix is a real symmetric matrix for which no principal minor determinant is less than any minor determinant formed from the same rows. Any open-circuit impedance matrix or short-circuit admittance matrix is known to be paramount, but for orders exceeding 3, it is not known whether any paramount matrix is realizable as such an impedance or admittance matrix. This paper does not answer the question, but does provide a sufficient condition.

A. S. Householder (Oak Ridge, Tenn.)

4723:

Schopf, Andreas H. On the Kantorovich inequality. Numer. Math. **2** (1960), 344–346.

The so-called Kantorovich inequality has received considerable attention recently [see, e.g., W. Greub and W. Rheinboldt, Proc. Amer. Math. Soc. **10** (1959), 407–415; MR **21** #3774; W. G. Strang, ibid. **11** (1960), 468; MR **22** #2904]. In this paper—which, after the author's untimely death, was organized and written by A. S. Householder—a very concise proof is presented for the following n-dimensional form of this inequality: Let A be a positive-definite hermitian matrix whose characteristic roots are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$. Let $\chi^2 = \lambda_1/\lambda_n$ and $\mu_\nu = \mu_\nu(x) = x^H A^\nu x$, where the superscript H represents conjugate transpose. Then $1 \leq \mu_{\nu+1}\mu_{\nu-1}/\mu_\nu^2 \leq (\chi + \chi^{-1})^2/4$ for all $x \neq 0$. In addition, certain interesting consequences of this theorem are discussed.

W. C. Rheinboldt (Syracuse, N.Y.)

4724:

Haas, Hans-Lothar. Bestimmung der Eigenwerte des allgemeinen Eigenwertproblems. Numer. Math. **2** (1960), 341–343.

The author discusses—rather summarily—a transformation of the general eigenvalue problem $(\lambda B+C)x=y$ into a special problem $(\lambda I_r+A)y=q$. Here A, B, C, I_r are $n\times n$ matrices, B may be singular (rank r) and I_r is diagonal with r "one's" on the main diagonal. The transformation is based on writing B in a form $B=UI_rD$, where U is lower- and D upper-triangular (possibly not ordered), and both these matrices are regular.

W. C. Rheinboldt (Syracuse, N.Y.)

4725:

Amir-Moéz, Ali R. Some equalities in a unitary space leading to equalities concerning singular values of sets of matrices. Math. Ann. **135** (1958), 388–390.

The author shows that certain simple equations concerning the norms of vectors in unitary space imply equations about singular values of matrices. Typical of the results is the following: Let A_i $(i=1, \cdots, k)$ have singular values $a_{i1}\geq\cdots\geq a_{in}$. If $A_i^*A_j=0$ $(i\neq j)$, $B=c_1A_1+\cdots+c_kA_k$ $(c_i>0, \sum c_i=1)$ has singular values $b_1\geq\cdots\geq b_n$, and if $B^*A_i=B^*A_j$ $(i\neq j)$, then $b_p^{-2}=a_{1p}^{-2}+\cdots+a_{kp}^{-2}$, whenever defined. Three other theorems in the same spirit are given. {Reviewer's note: The hypothesis that the coefficient a_i (in the author's notation) be positive is superfluous. Also, line 11 on p. 390 should read "$\lambda_p^2=\sum a_i^2\psi_{ip}^2$", requiring some obvious changes in the lines immediately preceding.}

A. J. Hoffman (New York)

4726:

Christiano, John G.; Dunbar, David L. Finding the characteristic roots of symmetric matrices. Boll. Un. Mat. Ital. (3) **14** (1959), 352–359.

The authors' nomenclature is misleading, since the real (or complex) matrices which they consider are those with identical columns, as well as those symmetric in the standard sense $(a_{ij}=a_{ji})$ which also have equal diagonal elements. For such 2 by 2 and 3 by 3 matrices, and for "partitioned combinations" of these of higher total dimension, the authors readily find similarity transformations yielding only zeros above the main diagonal; hence the eigenvalues with proper multiplicity may be read off as the diagonal elements.

F. H. Brownell (Seattle, Wash.)

4727:

Korganoff, A. Note sur la construction d'un repère orthogonal dont un des axes est un vecteur donné. Chiffres **3** (1960), 117–120. (English, German and Russian summaries)

It is proposed to determine an orthogonal matrix with prescribed first column and with $a_{i,j}=0$ for $j>i+1$.

H. Schwerdtfeger (Montreal)

4728:

Stander, J. W.; Wiegmann, N. A. Canonical forms for certain matrices under unitary congruence. Canad. J. Math. **12** (1960), 438–446.

The following theorems are proved. (1) Every complex skew-symmetric matrix is unitarily congruent to a direct sum of a suitable zero matrix and 2 by 2 real matrices of the form $\begin{bmatrix}0 & a\\ -a & 0\end{bmatrix}$, $a>0$. (2) If A is a *-symmetric quaternion matrix then there exists a quaternion unitary matrix U such that UAU^* is a real diagonal matrix with non-negative elements. (3) If A is a *-skew-symmetric quaternion matrix there exists a quaternion unitary matrix V such that VAV^* is a direct sum of a suitable zero matrix, 1 by 1 matrices of the form rji and $-rji$, $r>0$, and 2 by 2 matrices of the form $\begin{bmatrix}sji & t\\ -t & -sji\end{bmatrix}$, where $t>0$ and s is real. In (2) and (3), for any quaternion matrix M, M^* is the matrix obtained from M^T (the transpose of M) by changing the sign of the quaternion unit ij. M is called *-symmetric if $M=M^*$ and *-skew-symmetric if $M=-M^*$. On the other hand, M is called unitary if $MM^{CT}=M^{CT}M=I$, where M^C is obtained by replacing each element of M by its conjugate. The proofs, which make use of polar factorization, are essentially alike and leave one wishing for a unified treatment.

H. K. Farahat (Sheffield)

4729:

McCarthy, P. J. Note on Hermitian matrices over non-commutative valuation rings. J. London Math. Soc. **35** (1960), 1–4.

Suppose that D is a division ring with a valuation in a linearly ordered group whose corresponding valuation ring is R. Furthermore, assume that D has an anti-automorphism $a\to a^C$; also denote by A^T the transpose of a square matrix A with elements in R. The author calls A hermitian if $(A^C)^T=A$. He extends a theorem of Pall [Bull. Amer. Soc. **51** (1945), 889–893; MR **7**, 358] showing that the congruence of $\begin{pmatrix}a & 0\\ 0 & A\end{pmatrix}$ and $\begin{pmatrix}a & 0\\ 0 & B\end{pmatrix}$, $a=a^C$, implies, for hermitian matrices A, B, the congruence of A and B, if 2 is a unit in R. Here congruence is defined by $P^{CT}AP=B$, P a non-singular matrix with elements in R, and where A and B have essentially the same determinant in the sense of Dieudonné for not necessarily commutative rings. This result, using linear transformations as in the classical cases, is shown to imply that a non-singular hermitian matrix, as defined above, is congruent to a diagonal matrix.

O. F. G. Schilling (Chicago, Ill.)

4730:

Delève, N. Plan pseudo-hermitien. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 960–965.

A pseudo-hermitian plane is by definition a vector plane over the complex field in which the scalar product of vectors z_1 and z_2 of respective components x_1, y_1 and x_2, y_2 is

$$(z_1, z_2) = x_1^*x_2-y_1^*y_2,$$

the asterisk denoting the complex conjugate. If μ denotes the matrix diag $(1, -1)$, this is $(z_1, z_2)=z_1^+\mu z_2$, where z_1, z_2 now represent column-vectors and $^+$ the transposed conjugate. Every linear transformation in the plane is represented by a 2×2 matrix. Such a matrix A is called pseudo-hermitian if $A=\mu A^+\mu$, and a matrix U is called pseudo-unitary if $\mu U^+\mu U=I$. Canonical forms are found for pseudo-hermitian and pseudo-unitary matrices, and it is shown that changes of basis in the pseudo-hermitian plane are spinor transformations associated with proper rotations in Minkowski 'plane-time', that is, a plane in which the metric is given by $x_1^2+x_2^2+x_3^2$, where x_1, x_2 are real and x_3 is imaginary.

H. S. Ruse (Leeds)

4731:

Arrighi, Gino. Sulle matrici radice m^s della matrice nulla. Boll. Un. Mat. Ital. (3) **15** (1960), 128–133.

Author's summary: "Presento un semplice schema per la costruzione delle matrici che sono radice m^s della matrice nulla; con corredo di osservazioni ed esempio."

4732:

Ryser, H. J. Inequalities of compound and induced matrices with applications to combinatorial analysis. Illinois J. Math. **2** (1958), 240–253.

The author studies inequalities involving the elementary symmetric functions and the homogeneous product sums of the characteristic roots of a non-negative hermitian matrix. Applications are made to incidence matrices of v, k, λ configurations [see Chowla and Ryser, Canadian J. Math. **2** (1950), 93–99; MR **11**, 306], and the author's Theorem 3 on maximal determinants [ibid. **8** (1956), 245–249; MR **18**, 105] is obtained as a special case.

S. Chowla (Boulder, Colo.)

4733:

Wendelin, Hermann. Über iterierte s-te Ableitungen und s-te Adjungierte einer Matrix. J. Reine Angew. Math. **202** (1959), 137–152.

Let A denote a regular $n \times n$ matrix, $A^{(s)}$ the sth compound or derived matrix of A and $A^{[s]}$ the sth adjugate compound of A. Now iterate:

$$A^{(s_1,\cdots,s_k)} = (A^{(s_1,\cdots,s_{k-1})})^{(s_k)},$$

$$A^{(-1)^t(s_1,\cdots,s_k)} = (A^{(s_1,\cdots,s_k)})^{(-1)^t},\ A^{(-s)} = (A^{(s)})^{-1},$$

and define $A^{[s_1,\cdots,s_k]}$ similarly. A number of formulae relating $A^{(s_1,\cdots,s_k)}$ and $A^{[s_1,\cdots,s_k]}$ are proved, a typical one being

$$A^{[s_1,\cdots,s_k]} = |A|^{\sigma_{1,k}} A^{(-1)^k(s_1,\cdots,s_k)},$$

where

$$\sigma_{1,k} = \sum_{\kappa=0}^{k-1} (-1)^\kappa \prod_{\lambda=1}^{k-1-\kappa} \binom{n_\lambda - 1}{s_\lambda} \cdot \prod_{\mu=(k+1)-\kappa}^{k} s_\mu$$

and $n_\kappa = \binom{n_\kappa - 1}{s_\kappa}$ for $\kappa = 1, 2, \cdots, k$ and $n_0 = n$.

H. S. A. Potter (Aberdeen)

4734:

Gáspár, Julius. Eine axiomatische Theorie der Dieudonnéschen Determinanten. Publ. Math. Debrecen **6** (1959), 298–302.

Let K^* be the multiplicative group of a division ring K, K^*/C the factor group with respect to the commutator subgroup C, and φ_1 the canonical mapping of K^* on K^*/C. Let M_n be the multiplicative group of invertible matrices of order n with elements in K. The author shows that if φ is a mapping of M_n into K^*/C satisfying the two conditions (i) $\varphi(AB) = \varphi(A)\varphi(B)$ and (ii) $\varphi(A^{(t)}) = \varphi_1(t)\varphi(A)$, where $A^{(t)}$ is obtained from the matrix A by multiplying a row by $t \in K^*$, then $\varphi(A)$ is the Dieudonné determinant of A [Bull. Soc. Math. France **71** (1943), 27–45; MR **7**, 3].

B. N. Moyls (Vancouver, B.C.)

4735:

Ionescu, Gh. D. Nouvelle méthode de démonstration de la règle de Cramer. Inst. Politehn. Cluj. Lucrări Şti. **1959**, 79–81. (Romanian. Russian and French summaries)

4736:

Pethö, Árpád. Einige Bemerkungen zur eindeutigen Lösbarkeit von linearen algebraischen Gleichungssystemen. Magyar Tud. Akad. Mat. Kutató Int. Közl. **3** (1958), no. 1/2, 101–108. (Hungarian. Russian and German summaries)

The following theorem is proved regarding the unique solution of a system of linear equations: Let $A = A_{1,2,\cdots,n+1}$ be an m by $n+1$ matrix with columns $A_1, \cdots, A_{n+1}$ and let X be a column vector with components $x_1, \cdots, x_n, -1$. Then the system $AX = 0$ has a solution in which the variable x_k is determined uniquely if and only if (1) the rank of the matrix of the inhomogeneous system equals the rank of the homogeneous system, i.e., $\rho(A_{1,2,\cdots,n+1}) = \rho(A_{1,2,\cdots,n})$ and (2) the rank of the homogeneous system is one greater than the rank of the homogeneous system in which the kth column is dropped, i.e.,

$$\rho(A_{1,2,\cdots,n}) = \rho(A_{1,2,\cdots,k-1,k+1,\cdots,n}) + 1.$$

H. W. Kuhn (Princeton, N.J.)

4737:

Frăţilă, E. La généralisation de l'algorithme de Gauss et de l'identité de Sylvester. Inst. Politehn. Cluj. Lucrări Şti. **1959**, 61–68. (Romanian. Russian and French summaries)

The author begins with the Gauss reduction algorithm, which consists of the transformation of the system (I) $\sum_{j=1}^{n} a_{ij}x_j = b_i$, $i = 1, 2, \cdots, n$, into the system (II) $\sum_{j=1}^{n} A_{ij}x_j = B_i$, $i = 1, 2, \cdots, n$, with $A_{ij} = 0$ if $i > j$. The method is generalized as follows. One reduces the first $p_1 < n$ unknowns of the last $n - p_1$ equations of (I). In the system thus obtained, one reduces the first p_2 unknowns of the last $n - p_2$ equations, etc. Explicit formulas are given for the coefficients $a_{ij}^{(s)}$ of the transformed equations. The identity of Sylvester is generalized to the system of new coefficients $a_{ij}^{(s)}$.

E. Frank (Chicago, Ill.)

4738:

Abian, Smbat; Mendelson, Elliott. On the solution of an arbitrary system of linear equations. Math. Ann. **140** (1960), 245–248.

The authors give a new proof of a theorem of S. Gacsályi [Publ. Math. Debrecen **2** (1952), 292–296; MR **15**, 775], according to which an arbitrary system of linear equations over a skew field F has a solution in F if and only if it is consistent (in other terminology: compatible). Some generalized notions of consistency and solvability are given and their relationships exhibited.

A. Kertész (Debrecen)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 4729.

4739:

Posner, Edward C. Differentiably simple rings. Proc. Amer. Math. Soc. **11** (1960), 337–343.

Let R be a ring and D a set of derivations in R. R is called (D-) differentiably simple if $R^2 \neq 0$ and R has no proper two-sided ideal closed under D. In analogy to ordinary simple rings, such an R is shown, firstly, to

satisfy $R^2=R$, to have no absolute zero divisors and to possess what is justly called its differential centroid, a subfield of the ordinary centroid. R is further proved to be primary, and indeed prime if its differential centroid is of characteristic 0. If R is commutative, it has always a unit and is differentially closed in its quotient rings. Also the subring $B_D(R)$ of the endomorphism ring of R generated by multiplications and derivations in D is considered in connection with tensor products and differential polynomial rings. The commutative case is further studied, showing, e.g., that $B_D(R)$ is then simple.

T. Nakayama (Nagoya)

4740:

Pollak, Barth. The equation $lat=b$ in a quaternion algebra. Duke Math. J. **27** (1960), 261–271.

Let Q be a quaternion algebra over a field k of characteristic different from 2. If $x\in Q$, the trace and norm of x are denoted by Sx and Nx respectively, and if $Sx=0$, x is called pure. The author considers solutions of

$$\text{(H)}\qquad lat=b,\quad Nt=\sigma$$

for t in Q, when a and b are pure elements of Q, $Nb=\sigma^2Na\neq 0$ with $\sigma\in k$.

He defines K to be the kernel of the linear map $xT=\sigma ax-xb$ for all x in Q and shows that the dimension of K is 2. He proves Theorem 1: Let $\{t_1, t_2\}$ be a basis of K; then (H) has a solution t if and only if the binary quadratic form

$$B=\mu^2Nt_1+\mu\nu S(t_1t_2)+\nu^2Nt_2$$

represents σ in k. If μ, ν is a representation of σ by B, $t=\mu t_1+\nu t_2$ solves (H) and, conversely, every solution of (H) is of this form.

An alternative formulation of this result is given using Witt's theory of quadratic forms. If k is complete with respect to a discrete non-Archimedian valuation with finite residue class field k, the author proves the following conjecture of Pall: If Q splits, then (H) and (H′) are both solvable or both unsolvable, where (H′) is obtained from (H) by replacing σ by $-\sigma$. If Q is a division algebra, then exactly one of (H) and (H′) is solvable.

The author shows that if k is a number field, $k_\mathfrak{p}$ its completion at the prime $\mathfrak{p}$, and $Q_\mathfrak{p}=Q\otimes k_\mathfrak{p}$ is the scalar extension of Q, then (H) is solvable if and only if (H) is solvable over $Q_\mathfrak{p}$ for all primes $\mathfrak{p}$. Also, if $Na\neq 0$, then (H) is not solvable in $Q_\mathfrak{p}$ for a finite even number of primes.

The author also treats analogous questions for a maximal order M of a quaternion algebra.

B. W. Jones (Boulder, Colo.)

4741:

Rinehart, R. F. The equation $e^xe^y=e^{x+y}$ in quaternions. Rend. Circ. Mat. Palermo (2) **8** (1959), 160–162.

The author gives a simple geometric proof of the theorem by Morinaga and Nôno [J. Sci. Hiroshima Univ. Ser. A **17** (1954), 345–358; MR **16**, 558] on non-commutative quaternion solutions x, y of $e^xe^y=e^{x+y}$.

B. N. Moyls (Vancouver, B.C.)

4742:

Gerstenhaber, Murray. On nilalgebras and linear varieties of nilpotent matrices. II. Duke Math. J. **27** (1960), 21–31.

[For part I see Amer. J. Math. **80** (1958), 614–622; MR **20** #3161.] Let $\mathfrak{A}$ be an algebra (not necessarily finite-dimensional or associative) over a field F of characteristic zero. An element $a\in\mathfrak{A}$ is nil of index t if all possible products of t factors, each of them equal to a, vanish. Let R_a denote the linear transformation of $\mathfrak{A}$ defined by right multiplication: $xR_a=xa$; then it is shown that: (1) if $\mathfrak{A}$ is commutative nil of bounded index t, then $R_a^{2t-3}=0$; (2) if $\mathfrak{A}$ is power associative and $a\in\mathfrak{A}$ satisfies $a^t=0$, then the algebra generated by R_a, R_{a^2}, $\cdots$, is nilpotent of index $\leq 2(t-1)^2+1$. To obtain these results the author develops some properties of the ring of all operators $\{\delta(\alpha_1, \cdots, \alpha_n); \alpha_i\in F[x, y]\}$ acting on the free non-associative ring $F[x, y]$ generated by two elements x, y. For $f(x)\in F[x, y]$, $f^{(k)}(x)\delta(\alpha_1, \cdots, \alpha_n)$ is the sum of the elements obtained by all possible replacements of n of the k identical arguments x in f by $\alpha_1, \cdots, \alpha_n$ $(k\geq n)$.

S. A. Amitsur (Jerusalem)

4743:

Gerstenhaber, Murray. On nilalgebras and linear varieties of nilpotent matrices. III. Ann. of Math. (2) **70** (1959), 167–205.

A nilvariety V is a linear subspace of the algebra of all $n\times n$ matrices over a field K (which contains at least $\rho+1$ elements) such that all the matrices of V are nilpotent. The nilvariety is of index $\leq\rho$ if all its matrices satisfy $A^\rho=0$. The important result of this paper is that if V is a nilvariety of index $\leq\rho+1$ then $\dim V\leq\frac{1}{2}[n(n-1)-(n-\rho)(n-\rho-1)]$. The equality actually holds for the algebra $\mathfrak{A}_{\rho\mu}$ which contains all matrices (A_{ij}) with $A_{ij}=0$ for $i\geq\mu$ and $j\leq n-\rho+\mu+1$ $(1\leq\mu\leq\rho+1\leq n)$. Furthermore, the nilvarieties for which the equality holds are similar to one and only one of the algebras $\mathfrak{A}_{\rho\mu}$ (with the exception $A_{11}=A_{12}$ which is the zero algebra). The case $\rho=n-1$ yields the algebra $\mathbf{T}_n$ of all strictly triangular matrices. An interesting corollary is that the nilalgebra $\mathbf{T}_n/\mathbf{T}_n^{\rho+1}$, for $2\leq\rho\leq n-2$, cannot be faithfully represented by $n\times n$ matrices of rank $\leq\rho$. The main tool in the proof is a close study of the linear spaces obtained from V in the following way: Let $A_1, \cdots, A_k$ be a base for V and let $A_\sigma=(A_\sigma)_{ij}$. For a pair $\sigma=(ij)$, consider the vector $l_\sigma=((A_1)_\sigma, \cdots, (A_k)_\sigma)$. Choosing sets $S=\{\sigma_1, \cdots, \sigma_m\}$, one considers the linear space L_S generated by the vectors $l_{\sigma_1}, \cdots, l_{\sigma_m}$. The object is to determine standard sets S with the property $\dim L_S=\dim V$ and such that all the vectors l_σ $(\sigma\in S)$ are linearly independent. The first part of the paper introduces a notion of dominance of matrices: Let A and B be two $n\times n$ matrices in an algebraically closed field $\Omega\supset K$; consider the matrices as points in an n^2-space; then $A\succ B$ (A dominates B) if there exists a non-singular matrix Z such that B is a specialization of $Z^{-1}AZ$ over the base field K. It is shown that $A\succ B$ and $B\succ A$ if and only if A is similar to B, and that a necessary and sufficient condition for $A\succ B$ is that $\operatorname{rank}(A-\lambda)^i\geq\operatorname{rank}(B-\lambda)^i$ for all i and for every proper value λ of B.

S. A. Amitsur (Jerusalem)

4744:

Flanders, Harley. The meaning of the form calculus in classical ideal theory. Trans. Amer. Math. Soc. **95** (1960), 92–100.

Let $\mathfrak{o}$ be a Dedekind domain and k the quotient field of $\mathfrak{o}$. Let $x=(x_1, x_2, \cdots)$ be a set of indeterminates over k. The content of a rational function $r(x)$ in $k(x)$ is defined by generalizing the classical notion of the content of a poly-

nomial in $k[x]$. Let $\mathfrak{o}_x$ denote the set of all rational functions in $k(x)$ whose contents are contained in $\mathfrak{o}$. Then $\mathfrak{o}_x$ is a subring of $k(x)$, integrally closed in the quotient field $k(x)$. The author studies the relation between $\mathfrak{o}$-ideals in k and $\mathfrak{o}_x$-ideals in $k(x)$, and clarifies the meaning of the form calculus in classical ideal theory developed by Kronecker. Some of the main results are as follows: (1) $\mathfrak{o}_x$ is a principal ideal ring; (2) let $\bar{\mathfrak{a}}$ be any ideal of $k(x)$ and $\mathfrak{a}$ the g.c.d. of the contents of elements in $\bar{\mathfrak{a}}$; then $\mathfrak{a}=\bar{\mathfrak{a}}\cap k$ and $\bar{\mathfrak{a}}=\mathfrak{o}_x\mathfrak{a}$; the map $\bar{\mathfrak{a}}\to\mathfrak{a}$ defines an isomorphism of the ideal group of $k(x)$ onto the ideal group of k; the semi-group of integral ideals of $\mathfrak{o}_x$ is mapped onto that of $\mathfrak{o}$, prime ideals being preserved; (3) let K be a finite extension of k; let $\overline{\mathfrak{A}}$ be an ideal of $K(x)$ and $\mathfrak{A}$ the corresponding ideal of K; then $N_{K/k}\mathfrak{A}$ is the ideal of k corresponding to $N_{K(x)/k(x)}\overline{\mathfrak{A}}$. Similarly, various other properties of ideals in K relative to the extension K/k carry over to the identical properties of the corresponding ideals in $K(x)$ relative to the extension $K(x)/k(x)$.

K. Iwasawa (Cambridge, Mass.)

4745:

Pollak, G. On types of euclidean norms. Acta Sci. Math. Szeged **20** (1959), 252–268. (Russian)

The author calls a domain of integrity R a euclidean ring if there exists a mapping φ of R onto the set of ordinals less than some ordinal τ such that (a) $\varphi(ab)\geqq\varphi(b)$ for all $a, b\in R$, $a\neq 0$, and (b) for any pair $a, b\in R$, with $a\neq 0$, there exists a $q\in R$ satisfying $\varphi(b-aq)<\varphi(a)$. Then φ is called a euclidean norm and τ the type of φ. Theorems: (1) The types of euclidean norms of the ring of rational integers are the ordinals ω, λ $(1\leqq\lambda\leqq\omega)$. (2) Let R_j $(j=1,\cdots,5)$ denote the euclidean rings of algebraic integers in the complex fields of degree 2 over the rationals; they have the discriminants -3, -4, -7, -8, -11 [Th. Motzkin, Bull. Amer. Math. Soc. **55** (1949), 1142–1146; MR **11**, 311]. The types of euclidean norms in R_j are the limit ordinals $\leqq\omega^{n_j}$, where $n_j=3, 2, 1, 1, 1$, respectively. (3) If K is a field of infinite cardinality $\mathfrak{m}$, then the types of euclidean norms of $K[x]$ are the ordinals which are cofinal with ω and are of power $\leqq\mathfrak{m}$. If K is finite, then the norms are all of type ω.

L. Fuchs (Budapest)

4746:

Rédei, Ladislaus. Die einstufig nichtregulären Ringe. Acta Sci. Math. Szeged. **20** (1959), 238–244.

A single-step ("einstufig") non-regular ring is a non-regular ring all of whose pure subgroups are regular. The paper shows that the only such rings are: (1) zero rings of prime order and (2) direct sums of two finite fields. In showing that there are no infinite single-step non-regular rings with one generator the author makes repeated use of the following lemma: Let J be the ring of rational integers and let A be an ideal of $J[x]$; if $f(x)$, $g(x)\in xJ[x]$, $f(x)$, $g(x)\notin xA$ and $f(x)g(x)\in xA$ then there exist polynomials $F(x)$ and $G(x)$ in $xJ[x]$ such that $F(f(x))+G(g(x))\equiv x \pmod{xA}$. It is then shown that there exist no infinite single-step non-regular rings with more than one generator. *Drury W. Wall* (Iowa City, Iowa)

4747:

Cohn, Paul M. On the free product of associative rings. II. The case of (skew) fields. Math. Z. **73** (1960), 433–456.

This is an extension of the author's earlier work on the same subject [Math. Z. **71** (1959), 380–398; MR **21** #5648]. In this paper special consideration is given to the study of free products of (skew) fields. In the presence of suitable conditions guaranteeing the existence of free products, the problem of determining when a free product of rings has no zero divisors reduces to the case where the ground ring is a field K. In this case, it is shown that the free product of any family of K-rings without zero divisors is again without zero divisors. For the free product of two fields over a common ground field, arithmetical questions concerning units and prime decomposition are studied. It is shown that in this case any two principal left ideals with non-zero intersection together generate a principal left ideal. *W. E. Jenner* (Lewisburg, Pa.)

4748:

Tachikawa, Hiroyuki. A note on algebras of unbounded representation type. Proc. Japan Acad. **36** (1960), 59–61.

In generalization of Jan's [Ann. of Math. (2) **66** (1957), 418–429; MR **19**, 526] result, by making it applicable to the case of arbitrary ground field, the author shows that an algebra with an infinite two-sided ideal lattice is of unbounded type. The proof is a modification of Brummund's [Inaugural-Dissertation, Münster, 1939].

T. Nakayama (Nagoya)

4749:

Behrens, Ernst-August. Distributiv darstellbare Ringe. Math. Z. **73** (1960), 409–432.

The author continues his study of the distributivity of the lattice $V(\mathfrak{m})$ of all $\mathfrak{R}$-submodules of a left $\mathfrak{R}$-module $\mathfrak{m}$ for a semi-primary ring $\mathfrak{R}$ [Arch. Math. **8** (1957), 265–273; MR **19**, 939]. When $\mathfrak{m}$ is faithful and $V(\mathfrak{m})$ is distributive, $\mathfrak{m}$ is called a distributive representation of $\mathfrak{R}$. If $\mathfrak{R}=\mathfrak{R}_1+\cdots+\mathfrak{R}_q+\mathfrak{R}$ is a supplementary sum in which each $\mathfrak{R}_i$ is primary with radical $\mathfrak{P}_i$ of exponent e_i and $\mathfrak{R}\leqq\mathfrak{P}$ (the radical of $\mathfrak{R}$), then $\mathfrak{m}$ is a distributive representation of $\mathfrak{R}$ if and only if the (lattice) dimension of $V(\mathfrak{m})$ is $e_1+\cdots+e_q$. Every distributive representation of $\mathfrak{R}$ is cyclic and is isomorphic to a difference module $\mathfrak{R}-\mathfrak{l}$ for some left ideal $\mathfrak{l}$ of $\mathfrak{R}$. However, it is possible that two distributive representations of $\mathfrak{R}$ be not isomorphic. The author supplies a criterion for isomorphism of two distributive representations of $\mathfrak{R}$.

The paper includes a detailed examination of the case in which $\mathfrak{R}$ is primary. Here the notion of distributive representation is narrowed by requiring that $\mathfrak{m}$ be also a right K-module, where K is the division ring associated with $\mathfrak{R}$ via Wedderburn's theorem. The results of this discussion are too involved to be quoted here.

The paper concludes with some general observations relative to the class Δ of semi-primary rings which possess distributive representations. The class Δ is closed under the operations of forming (finite) direct sums and (finite) matrix rings as well as the operation of projection onto a direct summand; but Δ is not closed under the operation of taking homomorphic images. Finally a "radical" is introduced whose vanishing is equivalent to the possibility of expressing $\mathfrak{R}$ as a subdirect sum of rings in the class Δ.

M. F. Smiley (Riverside, Calif.)

4750:

Walter, John H. On the Galois theory of division rings. Proc. Amer. Math. Soc. **10** (1959), 898–907.

The author improves recent results of T. Nagahara and H. Tominaga [Math. J. Okayama Univ. **6** (1956), 1–21; **7** (1957), 169–172; MR **19**, 8; **20** #3894] and N. Nobusawa [ibid. **7** (1957), 179–183; MR **20** #1697]. His main result is as follows. Let K be a division ring, L a Galois division subring. On the group $G(K/L)$ place the finite topology. Call a group of automorphisms regular if it contains all inner automorphisms generated by the centralizer of its fixed division subring. Let $V_K(L')$ denote the centralizer of L' in K. Let $H=V_K(V_K(L))$. Theorem 8: Let K/L be a Galois and locally finite extension and let $[K:H]\leqq\aleph_0$. Then there exists a one-to-one correspondence between the regular closed subgroups G of finite degree and the division subrings L' of finite degree of K containing L. Furthermore, $G(K/L(G))=G$ and $L(G(K/L'))=L'$.

G. Whaples (Bloomington, Ind.)

4751:

Nobusawa, Nobuo; Tominaga, Hisao. Some remarks on strictly Galois extensions of simple rings. Math. J. Okayama Univ. **9** (1959/60), 13–17.

Les auteurs se proposent de généraliser les théorèmes connus (Hilbert-Noether-Speiser et Witt) affirmant la nullité des groupes de cohomologie $H^1(K^*, \mathfrak{G})$ et $H^1(K, \mathfrak{G})$ où K est une extension galoisienne d'un corps L et $\mathfrak{G}$ son groupe de Galois. Soient R un anneau simple, $\mathfrak{G}$ un groupe fini d'automorphismes de R; on suppose que: (1) le sous-anneau $V(\mathfrak{G})$ de R engendré par les éléments définissant les automorphismes intérieurs qui appartiennent à $\mathfrak{G}$, est un anneau simple; (2) si S est le sous-anneau des éléments de R invariants par $\mathfrak{G}$, le degré $[R:S]$ est égal à l'ordre de $\mathfrak{G}$. Sous ces conditions, les auteurs prouvent: (I) Un 1-cocycle de $\mathfrak{G}$ à valeurs dans R (groupe additif) est un cobord (autrement dit, si $\sigma\to x_\sigma$ satisfait $x_\tau+x_\sigma{}^\tau=x_{\sigma\tau}$ pour σ, τ dans $\mathfrak{G}$, alors il existe $a\in R$ tel que $x_\sigma=a-a^\sigma$). (II) Un 1-cocycle de $\mathfrak{G}$ à valeurs dans le groupe des matrices inversibles d'ordre n sur R est un cobord (autrement dit, si les matrices $\mathbf{A}_\sigma$ sont telles que $\mathbf{A}_\sigma\mathbf{A}_\sigma{}^\tau=\mathbf{A}_{\sigma\tau}$, il existe une matrice inversible $\mathbf{B}$ telle que $\mathbf{A}_\sigma=\mathbf{B}^{-1}\mathbf{B}^\sigma$). Enfin, ils étudient la structure de R lorsque le groupe $\mathfrak{G}$ est abélien.

J. Dieudonné (Paris)

4752:

Conrad, Paul. Generalized semigroup rings. II. Portugal Math. **18** (1959), 33–53.

This paper continues the study initiated in J. Indian Math. Soc. **21** (1957), 73–95 [MR **20** #887] of sets with a partial order and a partial multiplication obeying certain axioms. Embedding and structure theorems are proved.

H. A. Thurston (Vancouver, B.C.)

4753:

Iizuka, Kenzo; Nakahara, Isamu. A note on the semi-radical of a semiring. Kumamoto J. Sci. Ser. A **4**, 1–3 (1959).

It is shown that the Jacobson radical and the semi-radical of a semiring coincide, in denial of what is stated in Bourne and Zassenhaus [Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 907–914; MR **20** #3899].

T. Nakayama (Nagoya)

4754:

Bourne, Samuel. On compact semirings. Proc. Japan Acad. **35** (1959), 332–334.

Generalizing the finite case in a paper by the author and Zassenhaus [Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 907–914; MR **20** #3899], this paper shows that a compact semiring without zeroid is a ring. The proof uses Numakura's [Math. J. Okayama Univ. **1** (1952), 99–108; MR **14**, 18] theorem on compact semigroups with cancellation, but results in generalizing it to semigroups without zeroid.

T. Nakayama (Nagoya)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

See 4742.

HOMOLOGICAL ALGEBRA

4755:

Fujiwara, Tsuyoshi; Murata, Kentaro. Existence of derivations in graded algebras. J. Math. Soc. Japan **11** (1959), 85–93.

In the first part of this paper the results on the existence of derivations from a free graded algebra into an arbitrary graded algebra to be found in Chevalley's book, *The construction and study of certain important algebras* [Math. Soc. Japan, Tokyo, 1955; MR **17**, 342], are extended to the case of φ-derivations, and some of Chevalley's restrictions on degrees are slightly weakened. The next section deals with the problem of lifting derivations of graded algebras, and in the last section this problem is solved by giving a criterion which is a far-reaching extension, to graded algebras, of a result by A. Weil [*Foundations of algebraic geometry*, Amer. Math. Soc., New York, 1946; MR **9**, 303; p. 12].

A. Rosenberg (Evanston, Ill.)

GROUPS AND GENERALIZATIONS

See also 4639, 4752.

4756:

Higman, Graham. Some remarks on varieties of groups. Quart. J. Math. Oxford Ser. (2) **10** (1959), 165–178.

A variety of groups is a class of groups satisfying a given set of identical relations. A group G belongs to the variety $\mathfrak{B}$ generated by a class of groups $\mathfrak{X}$ if it is a homomorphic image of a subgroup of a direct product of factors isomorphic to groups in $\mathfrak{X}$. Using the commutator symbol $(x, y)=x^{-1}y^{-1}xy$, it is first shown that a group satisfies the identical relation $((x, y), (x^{-1}, y))=1$ if and only if all its two-generator subgroups are metabelian. Next it is shown that if $\mathfrak{B}$ is a variety whose identical relations are finitely based, then so also is the variety $\mathfrak{U}\circ\mathfrak{B}$, consisting of groups with normal subgroup in $\mathfrak{U}$ and factor group in $\mathfrak{B}$, if $\mathfrak{U}$ denotes a nilpotent variety (every group of which is nilpotent). Any variety $\mathfrak{B}$ is included between the smallest and largest varieties $\mathfrak{B}_n$ and $\mathfrak{B}^n$ having the same

n-generator groups as $\mathfrak{B}: \mathfrak{B}_1 \subset \cdots \subset \mathfrak{B}_n \subset \cdots \subset \mathfrak{B} \subset \cdots \subset \mathfrak{B}^n \subset \cdots \subset \mathfrak{B}^1$; it is determined by its n-generator groups for some n if either the ascending chain $\mathfrak{B}_n$ or the descending chain $\mathfrak{B}^n$ is finite. The author shows that the ascending chain of varieties $\mathfrak{B}_n$ (each generated by the n-generator groups of $\mathfrak{B}$) is properly infinite if $\mathfrak{B} = \mathfrak{A} \circ \mathfrak{B}$, where $\mathfrak{A}$ is a variety of abelian groups of prime exponent p, and $\mathfrak{B}$ is any variety, containing at least one non-abelian group, in which finitely generated groups are finite and of order prime to p. Any group G in this variety $\mathfrak{B}$ has a normal subgroup N in $\mathfrak{A}$, that can be thought of as an additive group of a vector space over the p-element field F_p; and also has a complementary subgroup, isomorphic to G/N, that is representable by linear transformations of this vector space. It is proved that this variety is not generated by any finite set of groups; in particular, it is not generated by its n-generator subgroups for any finite n. *J. S. Frame* (E. Lansing, Mich.)

4757:

Duguid, A. M. A class of hyper-*FC*-groups. Pacific J. Math. **10** (1960), 117–120.

For each ordinal α, a group Q_α is constructed such that the upper *FC*-series (the *FC*-chain in the reviewer's sense [Canad. J. Math. **5** (1953), 498–511; MR **16**, 216], $\{1\} = H_0 \leq H_1 \leq \cdots$) terminates with H_α and such that each $H_{\beta+1}/H_\beta$ is infinite and centerless. $\{1\}$ is the first such group; for limit ordinals, the restricted direct product of the earlier examples will do; while for a non-limit ordinal $\beta = \alpha + 1$, the wreath product of the regular representation of Q_α with a representation of an infinite, centerless *FC*-group as a permutation group with special fixed point properties on the natural numbers gives $Q_{\alpha+1}$. The special properties are that the stabilizer of each natural number is of finite index; and, given a set which includes almost all the natural numbers, the cross-cut of the corresponding set of stabilizers is trivial. *F. Haimo* (St. Louis, Mo.)

4758:

Neumann, B. H. On amalgams of periodic groups. Proc. Roy. Soc. London Ser. A **255** (1960), 477–489.

It is well known that two given groups A and B can be isomorphically embedded in a group G so that their intersection is a prescribed subgroup $A \cap B = H$. If A and B are finite, then G may be taken to be finite [B. H. Neumann, Philos. Trans. Roy. Soc. London Ser. A **246** (1954), 503–554; MR **16**, 10]. The author here investigates the corresponding problem when A and B are locally finite, periodic, or of finite exponent. An example shows that when A and B are both locally finite and of finite exponent it may not even be possible to choose G periodic; thus some extra conditions are necessary. Denote by LF the statement 'if A and B are locally finite, then G may be taken locally finite'; similarly for P (periodic) and FE (of finite exponent). Typical results are: (1) (H central in A and B)$\Rightarrow$LF & P & FE; (2) (H a direct factor of A)$\Rightarrow$ LF & P & FE; (3) (H finite)$\Rightarrow$LF; (4) (H central in A)$\Rightarrow$ FE; (5) (H central and of countable index in A)$\Rightarrow$LF; (6) (H finite and central in A)$\Rightarrow$P.

P. J. Higgins (London)

4759:

Schenkman, Eugene. On the norm of a group. Illinois J. Math. **4** (1960), 150–152.

The norm N of a group G is defined to consist of those elements of G that commute with each of its subgroups, and hence transform each element into a power of itself. Aiming at simplifying and improving the results of L. T. Wos [same J. **2** (1958), 271–284; MR **21** #1339], the author proves that the norm of a group is in the second center of the group, and that the centralizer of the norm includes the commutator subgroup of the group.

J. S. Frame (E. Lansing, Mich.)

4760:

Szász, F. Reduktion eines gruppentheoretischen Problems von O. J. Schmidt. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 369–372. (Russian summary, unbound insert)

O. Yu. Schmidt asked what infinite groups have no infinite proper subgroups. Szélpál showed [Publ. Math. Debrecen **1** (1949), 63–64; MR **11**, 157] that Prüfer's C_{∞^p} groups are the only commutative groups with this property. The truth of the Burnside conjecture would let us omit "commutative". The author shows that commutativity may be replaced by any one among certain broader properties: (1) *FC* (no element has infinitely many conjugates)· (2) locally-finite *S* (*S* means that the elements of finite order form a group which is the direct product of its Sylow subgroups); (3) *RN* (there exist normal systems with Abelian factor groups); (4) *X* (every homomorphic image is an *X*; every member has a nontrivial normal subgroup). *J. L. Brenner* (Madison, Wis.)

4761:

Rokos, Pant. K. Des ordres des éléments de groupes non Abéliens. Prakt. Akad. Athēnōn **34** (1959), 167–178. (Greek. French summary)

A large part of the paper is devoted to recording known elementary results from group theory. Let $A_1, A_2, \cdots, A_n$ be subgroups of a group S and assume that

$$a_i\Big(\prod_{j=i+1}^{n} A_j\Big) = \Big(\prod_{j=i+1}^{n} A_j\Big)a_i$$

for every a_i in A_i and $i = 1, 2, \cdots, n-1$, where $\prod_{j=i+1}^{n} A_j$ is the complex of all elements of the form $a_{i+1}a_{i+2}\cdots a_n$, with a_j in A_j. Under these assumptions it is proved that $(a_ib)^k = a_i^k b_1$ for b, b_1 in $\prod_{j=i+1}^{n} A_j$. If in addition to the above hypotheses it is assumed that the intersection of A_i with $\prod_{j=i+1}^{n} A_j$ is the identity element for $i = 1, 2, \cdots, n-1$, then every $(a_ib)^k$ with a_i in A_i and b in $\prod_{j=i+1}^{n} A_j$ is in $\prod_{j=i+1}^{n} A_j$ if and only if the order of a_i divides k. If in addition to all above hypotheses we assume $A_ib = bA_i$ for each b in $\prod_{j=i+1}^{n} A_j$, then the order of $a = \prod_{i=1}^{n} a_i$ is the greatest common divisor $(t_1, t_2, \cdots, t_n)$, where t_i is the order of a_i. This is a generalization of the known result for abelian groups. *L. A. Kokoris* (Chicago, Ill.)

4762:

Nagai, Osamu. Supplement to note on Brauer's theorem of simple groups. II. Osaka Math. J. **11** (1959), 147–152.

This paper seeks to fill a gap in the proof of a theorem in a previous paper of the author [Osaka Math. J. **5** (1953), 227–232; MR **15**, 600]. The methods used are those of R. Brauer's theory of modular characters [cf. Ann. of Math. (2) **44** (1943), 57–79; MR **4**, 266].

H. K. Farahat (Sheffield)

4763:

Baumslag, Gilbert. Roots and wreath products. Proc. Cambridge Philos. Soc. **56** (1960), 109–117.

Let A and B be groups with certain existence and uniqueness properties for pth roots. This paper studies the problem of finding if and when these properties are also possessed by the wreath product of A by B.

Let p be a fixed prime. A group G is called an $\mathfrak{E}$-group if pth roots always exist in G, and a $\mathfrak{U}$-group if pth roots are unique. If a group is both an $\mathfrak{E}$-group and a $\mathfrak{U}$-group it is called an $\mathfrak{E}\mathfrak{U}$-group. The classes of all $\mathfrak{E}$-groups, all $\mathfrak{U}$-groups, and all $\mathfrak{E}\mathfrak{U}$-groups are designated by $\mathfrak{E}$, $\mathfrak{U}$, and $\mathfrak{E}\mathfrak{U}$ respectively. For definition of the wreath product of A by B (denoted by $A \wr B$) see the author, same Proc. **55** (1959), 224–231 [MR **21** #4179]. The unrestricted wreath product of A by B (denoted by $A \overline{\wr} B$) is defined similarly using the unrestricted direct product.

If $A \in \mathfrak{E}$ and $B \in \mathfrak{U}$ and B is periodic, then $A \wr B$ and $A \overline{\wr} B \in \mathfrak{E}$. If $A \neq 1$, then $A \wr B \in \mathfrak{E} \cap \mathfrak{U}$ if and only if B is periodic and $A, B \in \mathfrak{E} \cap \mathfrak{U}$. If $A, B \in \mathfrak{E}$ and $A \neq 1$, then $A \overline{\wr} B \in \mathfrak{E}$ if and only if B does not contain elements of order p. If $A, B \in \mathfrak{U}$, $A \neq 1$, then $A \overline{\wr} B \in \mathfrak{U}$ if and only if B is periodic. The proof uses results of the author which are to appear in Acta Math., and some results on extensions.

The article also includes a proof due to P. Hall: An extension of a ZA-group [see Kurosh, *The theory of groups Vol. II*, Chelsea, New York, 1956; MR **18**, 188; p. 218] in $\mathfrak{E}$ by a periodic group in $\mathfrak{U}$ is in $\mathfrak{E}$.

R. R. Struik (Vancouver, B.C.)

4764:

Zelmer, Violeta. Sur un sous-groupe base d'un groupe abélien primaire. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) **5** (1959), 1–4. (Romanian. Russian and French summaries)

A basic subgroup [Kulikov, Mat. Sb. (N.S.) **16 (58)** (1945), 129–162; MR **8**, 252] is constructed for the primary abelian group of Kuroš [ibid. **5 (47)** (1939), 347–354; MR **2**, 2] without elements of infinite height, indecomposable into a direct sum of cyclic groups and of continuum power. The basic subgroup is found by salvaging certain cyclic subgroups from the original process of Kuroš and forming their weak direct sum. The construction of Kuroš is presented in a misleading manner.

F. Haimo (St. Louis, Mo.)

4765:

Walker, Elbert A. Direct summands of direct products of abelian groups. Arch. Math. **11** (1960), 241–243.

Let G_α ($\alpha \in I$) be cotorsion groups in the sense of Harrison [Ann. of Math. (2) **69** (1959), 366–391; MR **21** #3481], i.e., the G_α are reduced abelian groups satisfying $\mathrm{Ext}(H, G_\alpha) = 0$ for all torsion-free abelian groups H. Then the direct product (= complete direct sum) G of the G_α has a maximal torsion-free direct summand H. $H = 0$ if and only if the factor group of G with respect to its torsion subgroup is divisible. Any non-zero torsion-free direct summand of G is uncountable.

L. Fuchs (Budapest)

4766:

Kolettis, George, Jr. Semi-complete primary abelian groups. Proc. Amer. Math. Soc. **11** (1960), 200–205.

A primary abelian group G is called semi-complete if it is a direct sum of a torsion-complete group H and a direct sum of cyclic groups. (Thus H is the torsion subgroup of its completion in the natural p-adic topology.) Let $G = H \oplus K = H' \oplus K'$ where H, H' are torsion-complete groups and K, K' are direct sums of cyclic groups. Then there exists an integer n such that $p^n H \cong p^n H'$ and $p^n K \cong p^n K'$. The two decompositions of G have isomorphic refinements.

L. Fuchs (Budapest)

4767:

Brahana, H. R. A new type of characteristic subgroup of prime-power groups. Canad. J. Math. **12** (1960), 477–482.

An investigation, motivated by geometrical considerations, of some characteristic subgroups of the groups of order p^{11} previously studied by the same author [Illinois J. Math. **2** (1958), 641–717; MR **21** #1344].

H. A. Thurston (Vancouver, B.C.)

4768:

Kochendörffer, Rudolf. Ein Satz über Sylowgruppen. Math. Nachr. **17**, 189–194 (1959).

Let a group $\mathfrak{G}$ with subgroup $\mathfrak{U}$ be written as the sum of its right cosets $\mathfrak{U}R$. If the system $\mathfrak{R}$ of coset representatives R has the property that $U^{-1}\mathfrak{R}U = R$ for each U from $\mathfrak{U}$, it is called distinguished (ausgezeichnet). The subgroup $\mathfrak{U}$ of $\mathfrak{G}$ is called normally completable ("ergänzbar") if $\mathfrak{G} = \mathfrak{U}\mathfrak{N}$ where $\mathfrak{N}$ is a normal subgroup of $\mathfrak{G}$ and $\mathfrak{U} \cap \mathfrak{N} = 1$. Clearly the elements of $\mathfrak{N}$ form a distinguished representative system for such a $\mathfrak{U}$. The author's main theorem is the following: A Sylow subgroup $\mathfrak{P}$ of a finite group $\mathfrak{G}$, whose commutator group is contained in its center, possesses a distinguished representative system if and only if it is normally completable.

J. S. Frame (E. Lansing, Mich.)

4769:

Gol'berg, P. A. S-radical and Sylow bases of infinite groups. Mat. Sb. (N.S.) **50 (92)** (1960), 25–42. (Russian)

The concepts defined and studied in this interesting article have their roots in P. Hall's paper [Proc. London Math. Soc. (2) **43** (1937), 316–323] and in subsequent papers by Černikov [Mat. Sb. (N.S.) **22 (64)** (1948), 101–133; MR **9**, 566], Čunihin [Dokl. Akad. Nauk SSSR **59** (1948), 443–445; MR **9**, 492], and the author. The group H is called an S-group if it is the direct product of its Sylow subgroups. {The definition of a special group H' in Kuroš' book [*The theory of groups*, Chelsea, New York, 1960; MR **22** #727] coincides with the above definition if H' is periodic.} Subgroups and homomorphic images of S-groups are S-groups. Every group G contains a normal S-subgroup $S(G)$ which is maximal; in fact, which contains every normal S-subgroup of G. $S(G)$ is called the S-radical of G; it contains the Plotkin radical [Trudy Moskov. Mat. Obšč. **6** (1957), 299–336; MR **19**, 529]. The converse inclusion relation is related to the Burnside conjecture. $S(G)$ is the direct product of the $Q^{(p)}$, where $Q^{(p)}$ is the intersection of all p-Sylow subgroups of G. An FS-group is one in which every Sylow subgroup has only a finite number of conjugates. If Π is a set of primes, a Π-FS-group K is similarly defined. Subgroups inherit this property. A homomorphic image need not; $K/S(K)$ does, and its p-Sylow subgroups are finite ($p \in \Pi$).

Let a set of Sylow subgroups H of G generate A; let each H have only a finite number of conjugates in G.

Then A is periodic. The periodic part of an FS-group G is a group; the periodic part of $G/S(G)$ is thin (slender) and layer-finite in $G/S(G)$. A group is called S-complete if it is generated by every Sylow set $\{S_p\}$, i.e., a set with one (Sylow) member corresponding to each pertinent prime. Finite groups and infinite periodic abelian groups are S-complete; some infinite periodic groups are not. Every periodic FS-group is S-complete.

A group G is Π-separable if it has an ascending normal series in which no two primes in Π appear as orders of two elements of the same factor group. [If G is a p-group, or a group without torsion, this definition is vacuously satisfied.] Subgroups inherit this property. Homomorphic images do also if G is periodic. Conversely, if H, G/H are Π-separable, so is G. A locally normal, locally Π-separable group is Π-separable.

A Π-separable Π-FS-group contains a Sylow Π-basis, but may contain two inconjugate ones. If an FS-group has a complete Sylow basis, it is separable. A locally Π-separable Π-FS-group is Π-separable; in particular, a locally solvable FS-group is separable, and has a complete Sylow basis. *J. L. Brenner* (Madison, Wis.)

4770:

Blackburn, Norman. Nilpotent groups in which the derived group has two generators. J. London Math. Soc. **35** (1960), 33–35.

Let G be a nilpotent group whose derived group G' can be generated by two elements. Then G' is either Abelian or a finite metacyclic group of class 2, and $[\gamma_2(G), \gamma_3(G)] = 1$. This has been proved for finite p-groups by the author [Proc. Cambridge Philos. Soc. **53** (1957), 19–27; MR **18**, 464], and the present proof proceeds by reduction to this special case. The crucial lemma for the reduction is the following. Let G be nilpotent of class k, and suppose that the p-component S of G has finite exponent p^m. Then there exists an integer α depending only on k and p such that $P_{\alpha+m}(G) \cap S = 1$, where $P_r(G)$ denotes the group generated by all p^rth powers of elements of G.

P. J. Higgins (London)

4771:

Baur, Heribert. Ein Kommutativitätskriterium für unendliche auflösbare Gruppen. Arch. Math. **11** (1960), 176–182.

A generalization of solubility is introduced as follows. Abelian groups are generalized-soluble of degree ("Stufe") 0. The group G is generalized-soluble of degree ρ if it possesses an invariant system whose factors are generalized soluble of degree $< \rho$. The principal result of the paper is the "Hauptsatz": If the group G is generalized-soluble of degree ρ and if all finite factors of G are abelian, then G is abelian. The hypothesis can be further weakened to assume generalized solubility of degree ρ only for all two-generator subgroups of G. The proof relies on a result first proved by the reviewer [Compositio Math. **13** (1956), 47–64; MR **19**, 632; Theorem 6.3]; the author is unaware of this fact and proves the result independently, but by a similar method. *B. H. Neumann* (Manchester)

4772:

Tyškevič, R. Linear nilpotent groups with Abelian commutator subgroup. Dokl. Akad. Nauk BSSR **2** (1958), 231–233. (Russian)

This article is a further step towards the goal of classifying irreducible nilpotent matrix groups, and ascertaining the number of classes of conjugates of them under the full linear group. See Suprunenko [Mat. Sb. (N.S.) **31 (73)** (1952), 353–358; Dokl. Akad. Nauk SSSR **99** (1954), 23–25; **102** (1955), 41–44; Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat. **1953**, no. 15, 3–6; Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 273–274; MR **14**, 447; **16**, 793; **17**, 456]. The following results are announced. First, detailed properties of the central series of a nilpotent $n \times n$ matrix group Γ of nilpotency class l are given; the underlying field P is algebraically closed. Theorem 6: From the construction used to define Γ it follows that, if p is a prime, GL (p^2, P) contains at least two inconjugate classes of irreducible nilpotent subgroups, maximal of class l (for each l). Theorem 4: If n is not divisible by a cube, the commutator subgroup of an irreducible nilpotent group from GL (n, P) is abelian. Theorem 5 gives a description of one member of each class of conjugates when the degree of nilpotency is 3. There are at least as many classes of conjugates as there are non-isomorphic abelian groups of order n. *J. L. Brenner* (Madison, Wis.)

4773:

Zacher, Giovanni. On lattice dual-homomorphisms between finite groups. Rend. Sem. Mat. Univ. Padova **30** (1960), 65–75.

Let G, $\bar{G}$ be groups and $L(G)$, $L(\bar{G})$ their subgroup lattices. A homomorphism of $L(G)$ onto the dual of $L(\bar{G})$ is called a dual-homomorphism of G onto $\bar{G}$. G is said to have a dual if it is dual-isomorphic to some $\bar{G}$. M. Suzuki [*Structure of a group and the structure of its lattice of subgroups*, Springer, Berlin, 1956; MR **18**, 715] has determined the finite soluble groups with duals. The author shows (a) that every finite group with a dual is a direct product, where each factor is a soluble or simple group with a dual and where the orders of the factors are relatively prime in pairs, and (b) that every dual-homomorphic image of a finite group has a dual. (The existence of non-abelian simple groups with a dual is an open question.) In neither part is the proof easy. The procedure in (b) is briefly as follows. If $\bar{G}$ is a dual-homomorphic image of G it is shown that the Frattini factor group $\bar{G}/\Phi(\bar{G})$ has a dual. The resulting direct decomposition implies one of $\bar{G}$ [Suzuki, loc. cit., p. 5] and this in turn, by a general result of the author, implies one of G/E, where E is the largest subgroup of G mapped onto $\bar{G}$ in the dual-homomorphism. The rest of the proof is fairly straightforward. Misprints (chiefly omission of bars over letters) are numerous. The sentence in l. 2, p. 68, should be, "Let N be a minimal normal non-abelian simple subgroup of G". In l. −6, p. 68, "normal soluble subgroups" seems to have been intended rather than "normal subgroups". The argument on p. 68, l. 16 et seq., is simplified by noting that every proper normal subgroup of a p-group P meets the centre of P non-trivially. *G. E. Wall* (Chicago, Ill.)

4774:

Tamaschke, Olaf. Ringtheoretische Behandlung einfach transitiver Permutationsgruppen. Math. Z. **73** (1960), 393–408.

The reduction of the permutation representation $\mathfrak{D}(\mathfrak{G})$ of a simply transitive finite group $\mathfrak{G}$ is treated from a ring theoretical view point, and a number of results of Schur, Wielandt, and Frame are derived using ring-theoretical techniques. These are further extended to the cases where the underlying field K has a prime characteristic p that either (II) divides the order g of $\mathfrak{G}$, or (III) divides the order g_0 of the chosen subgroup $\mathfrak{G}_0$ of $\mathfrak{G}$, where $\mathfrak{G}_0$ leaves fixed the first of the n symbols that $\mathfrak{G}$ permutes. To each subgroup $\mathfrak{G}_0$ of $\mathfrak{G}$ corresponds an idempotent ε in the group ring $\mathfrak{A}$ of $\mathfrak{G}$ over K, such that the right ideal $\varepsilon\mathfrak{A}$ forms a representation module for the permutation representation of $\mathfrak{G}$ induced by $\mathfrak{G}_0$. Except in case (III), when $p|g_0$, this ε is the ring element $(1/g_0)\sum G$, $G\in\mathfrak{G}_0$, called a "subgroup average" by Frame [Ann. of Math. (2) **55** (1952), 85–100; MR **13**, 530; p. 86ff]. The ring $\varepsilon\mathfrak{A}\varepsilon$, consisting of a linear combination of double cosets of $\mathfrak{G}_0$ in $\mathfrak{G}$, serves as a left multiplication domain for the right ideal $\mathfrak{A}$, and is anti-isomorphic to the $\mathfrak{A}$-endomorphism ring of $\varepsilon\mathfrak{A}$. By using an easily defined ring anti-isomorphic to $\varepsilon\mathfrak{A}\varepsilon$, based on inverse double cosets, the author obtains a ring-theoretical description of the double coset matrices that span the commuting ring of $\mathfrak{D}(\mathfrak{G})$. By resolving the idempotent ε into a direct sum of primitive idempotents, he derives expressions for the transitive factor and group characters. In a brief treatment of the modular case ($p|g$ or $p|g_0$), these expressions involve the Cartan invariants and the Brauer decomposition numbers, in a well known way. *J. S. Frame* (E. Lansing, Mich.)

4775:

Hoare, A. H. M. A note on 2-soluble groups. J. London Math. Soc. **35** (1960), 193–199.

This paper proves the inequality $e_p \geq 1+[\frac{1}{2}(l_p+1)]$ for 2-solvable groups. Here $p=2$, while l_p is the p-length and e_p is the exponent of a Sylow p-subgroup. The methods used are those of P. Hall and G. Higman [Proc. London Math. Soc. (3) **6** (1956), 1–42; MR **17**, 344], and the result fills a gap in the Hall and Higman paper. It is not claimed that this result is best possible.

Marshall Hall, Jr. (Pasadena, Calif.)

4776:

Ludwig, Karl. Direkte Produkte von n- und n'-Gruppen. Arch. Math. **11** (1960), 171–175.

Let n, n' denote complementary sets of primes. A finite group G is n-homogeneous [R. Baer, Illinois J. Math. **2** (1958), 619–640; MR **21** #3485] if, for every n-subgroup U of G, $[\mathfrak{N}U:\mathfrak{C}U]$ is an n-number, where $\mathfrak{N}U$, $\mathfrak{C}U$ denote respectively the normalizer and centralizer of U in G. In this paper G is called strictly n-homogeneous if, for every n-subgroup U of G, $[G:\mathfrak{C}U]$ is an n-number. (It is enough to consider maximal n-subgroups.) The following three conditions on G are shown to be equivalent: (a) G is the direct product of an n-group and an n'-group; (b) G is strictly n-homogeneous and all maximal n-subgroups of G have the same order; (c) every subgroup of G is strictly n-homogeneous. *P. J. Higgins* (London)

4777:

Mennicke, Jens. Einige endliche Gruppen mit drei Erzeugenden und drei Relationen. Arch. Math. **10** (1959), 409–418.

Let G be a finite group defined abstractly by a minimal set of n generators satisfying a minimal set of m relations. Since G is finite, $n\leq m$. The problem of finding finite groups for which $n=m$ is of some interest. It is known, for example, that for a finite group, $n=m$ if and only if the group has trivial multiplicator [I. Schur, J. Reine Angew. Math. **127** (1904), 20–50; B. H. Neumann, Publ. Math. Debrecen **4** (1956), 190–194; MR **18**, 12]. Particular groups with $m=n=2$ are known [H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups*, Springer, Berlin, 1957; MR **19**, 527]. In this paper the author investigates the group G^* defined by $bab^{-1}=a^{1+t}$, $cbc^{-1}=b^{1+t}$, $aca^{-1}=c^{1+t}$ $(t>1)$, shows that G^* is finite, that the subgroup $H=\{a^3, b^3, c^3\}$ is normal abelian, and that the factor group G^*/H is nilpotent, and finds the order of G^* and the class of G^*/H. The author indicates some generalizations (e.g., replacing the exponents $1+t$ by $1+t_1$, $1+t_2$, $1+t_3$) and conjectures that if G is finite and $n\geq 3$ then $\binom{n}{2}\leq m$. *W. Moser* (Winnipeg, Man.)

4778:

Hobby, Charles. A characteristic subgroup of a p-group. Pacific J. Math. **10** (1960), 853–858.

If G is a finite p-group, denote by $\theta(G)$ the subgroup generated by all elements $y^{-p}x^{-p}(xy)^p$. This subgroup is used to study p-groups in which the Frattini subgroup $\Phi(G)$ is equal to the subgroup $P(G)$ generated by all elements x^p. Theorem: Let $p\neq 2$, and let $x_1, x_2, \cdots, x_k$ be coset representatives of a minimal basis of G/G'. Then $P(G)=\Phi(G)$ if, and only if, G' is generated by suitable powers of $x_1, x_2, \cdots, x_k$. Further, in this case $P(G')=\Phi(G')=\theta(G)$.

Applications to metacyclic p-groups are given. For $p\neq 2$, each of the following conditions is necessary and sufficient for G to be metacyclic: (i) $G/\theta(G)$ is metacyclic; (ii) $G/\Phi(G')G_3$ is metacyclic; (iii) G can be generated by two elements and $P(G)=\Phi(G)$. Condition (ii) is due to N. Blackburn [Proc. Cambridge Philos. Soc. **54** (1958), 327–337; MR **21** #1348] and condition (iii) to B. Huppert [Math. Z. **58** (1953), 243–264; MR **14**, 1059]. They are deduced here from condition (i).

P. J. Higgins (London)

4779:

Higman, Graham. Enumerating p-groups. I. Inequalities. Proc. London Math. Soc. (3) **10** (1960), 24–30.

If the number of isomorphism-classes of groups of order p^n is p^{An^3}, where, of course, A depends on n and p, then

$$2/27-o(n)\leq A\leq 2/15+o(n).$$

This depends on the result (also proved in the present paper) that if the number of conjugate classes in a Sylow p-subgroup of $\mathrm{GL}_n(p)$ is p^{Bn^2}, then

$$1/12-o(n)\leq B\leq 1/4+o(n).$$

Two more papers are promised.

H. A. Thurston (Vancouver, B.C.)

4780:

Hobby, Charles. The Frattini subgroup of a p-group. Pacific J. Math. **10** (1960), 209–212.

In a p-group G, let H_1 denote any non-abelian subgroup

whose center is cyclic, and let H_2 denote any non-abelian subgroup the index of whose derived group is p^2. Paralleling the theorems of Burnside according to which neither H_1 nor H_2 can be the derived group of G, the author shows that neither subgroup can be the Frattini subgroup $\Phi(G)$ (i.e., the intersection of all maximal subgroups of G). The proofs hold equally well if $\Phi(G)$ is replaced by some characteristic subgroup $\psi(G)$ contained in $\Phi(G)$, such that $\psi(G/N)=\psi(G)/N$ for any normal subgroup N of G contained in $\Phi(G)$. They thus include Burnside's theorems for the derived groups. Lemmas of N. Blackburn [Proc. Cambridge Philos. Soc. **53** (1957), 19–27; MR **18**, 464] and of W. Gaschütz [Math. Z. **58** (1953), 160–170; MR **15**, 285] are used with two other lemmas as the basis for an inductive proof of theorem 1. Lemma 4 states that a non-abelian group H_1 of order p^3 cannot be Frattini subgroup of any p-group. *J. S. Frame* (E. Lansing, Mich.)

4781:

Wan, Zhe-xian. On the commutator subgroup of the orthogonal group. Sci. Record (N.S.) **3** (1959), 599–603.

Let F be a field of characteristic $\neq 2$, and let S be the $2n\times 2n$ matrix

$$\begin{pmatrix} O & I \\ I & O \end{pmatrix}.$$

Define the orthogonal group $O_{2n}=O_{2n}(F,S)$ to be the group of all matrices X such that $XSX'=S$, and let O_{2n}^+ be the subgroup for which $|X|=1$. Let Ω_{2n} be the commutator subgroup of O_{2n}. Each $X\in O_{2n}$ is expressible as

$$X=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I-J & J \\ J & I-J \end{pmatrix},$$

where $|A|\neq 0$ and where $J=J^2$ is diagonal. The author proves the following conjecture (due to L. K. Hua): The commutator subgroup Ω_{2n} consists of all such matrices $X\in O_{2n}$ for which $|A|$ is a square in F and J has even rank. Using this, the author gives a new proof of a result due to Eichler [*Quadratische Formen und orthogonale Gruppen*, Springer, Berlin, 1952; MR **14**, 540], namely, that O_{2n}^+/Ω_{2n} is isomorphic to the multiplicative group of non-zero elements of F modulo their squares. The proofs in this paper use only elementary calculations with matrices. *I. Reiner* (Urbana, Ill.)

4782:

Choe, Tae Ho. On a quasi-ordered group. Kyungpook Math. J. **2** (1959), 47–52.

Quasi-ordered groups are defined by omitting the anti-symmetric law from the postulates defining a partly ordered group. Various known properties of partly ordered groups are then generalized to quasi-ordered groups.

G. Birkhoff (Cambridge, Mass.)

4783:

Cristescu, Romulus. Sur les groupes dirigés. Czechoslovak Math. J. **10** (**85**) (1960), 17–26. (Russian summary)

Let G be a directed group and Q a directed subgroup. Q is a component of G if for any $x\in G_+$ (=the positive cone of G) there exist $x'\in Q_+$ and $x''\in Q_+{}^*$ satisfying $x=x'+x''$. Here $Q_+{}^*$ is the set of all $y\in G_+$ such that $\inf(x,y)=0$ for all $x\in Q_+$. By the aid of components direct decompositions of G are dealt with. *L. Fuchs* (Budapest)

4784:

Feit, Walter. On a class of doubly transitive permutation groups. Illinois J. Math. **4** (1960), 170–186.

This paper studies doubly transitive groups in which only the identity fixes three letters. Such a group G doubly transitive on $m+1$ letters will be of order $(m+1)mq$, where q divides $m-1$ and is the order of a subgroup fixing two letters. We have $q>1$ except for the nearfield groups classified by Zassenhaus [Abh. Math. Sem. Univ. Hamburg **11** (1935), 187–220]. G will contain a Frobenius subgroup H of order mq and H contains a normal regular subgroup M of order m. By a result of John Thompson's, M is nilpotent. The chief result of this paper is that under a certain assumption G either (1) is a triply transitive group or of index 2 in such a group or (2) contains a normal subgroup of order $m+1$. These groups are of known types. The assumption is that M is not a non-abelian p-group with $[M:M']<4q^2$. This last assumption is indeed necessary, for since the publication of this paper, Suzuki has found a new class of simple groups of order $(2^{4n+2}+1)2^{4n+2}(2^{2n+1}-1)$ of this kind in which $[M:M']=2^{2n+1}$ and $q=2^{2n+1}-1$. A large part of the paper is devoted to an extension of the theory of exceptional characters, and it is to be expected that this theory will have other applications.

Marshall Hall, Jr. (Pasadena, Calif.)

4785:

Feit, Walter. A characterization of the simple groups SL(2, 2^a). Amer. J. Math. **82** (1960), 281–300.

Let G be a finite group of even order such that the centralizer of any involution in a 2-Sylow subgroup S_2 of G is contained in the centralizer of S_2. It is proved that then at least one of the following is true: (I) S_2 is cyclic; (II) S_2 is normal in G; (III) G is the direct product of a group D of odd order and a group L isomorphic to SL(2, 2^a) with some $a>1$. In particular, G is isomorphic to SL(2, 2^a), $a>1$, in case it is simple and non-cyclic. Besides a theorem on the case $a=2$, it follows also that if the centralizer of every involution in a non-cyclic simple group with abelian 2-Sylow subgroups is nilpotent, the group is isomorphic to SL(2, 2^a), $a>1$. This refines the characterizations by Suzuki [Trans. Amer. Math. Soc. **92** (1959), 191–204; MR **21** #7252] and Brauer, Suzuki and Wall [Illinois J. Math. **2** (1958), 718–745; MR **21** #3487] of SL(2, 2^a), $a>1$. For the proof a group G of minimal order is considered, for which the theorem fails to hold, and after some preliminary considerations, which are analogous to those in Suzuki [loc. cit.], two cases are introduced according to whether the centralizer of an element in the centralizer of S_2, different from 1, is necessarily contained in the normalizer of S_2, or not. While the first case is settled, leading to a contradiction, by the use of a result in the author's recent paper in Illinois J. Math. **4** (1960), 170–186 [see preceding review], the treatment of the second case depends on some arithmetical properties of certain characters. *T. Nakayama* (Nagoya)

4786:

Specht, Wilhelm. Die Charaktere der symmetrischen Gruppe. Math. Z. **73** (1960), 312–329.

Let $\mathfrak{z}=(z_1,z_2,z_3,\cdots)$ be an infinite sequence of independent indeterminates; and if $\alpha=(\alpha_1,\alpha_2,\alpha_3,\cdots)$ is a sequence of non-negative integers (almost all zero), write

$\binom{\mathfrak{d}}{\mathfrak{a}} = \prod_1^\infty \binom{z_i}{a_i}$, and $\pi_q(\mathfrak{d}) = \sum \binom{\mathfrak{d}}{\mathfrak{a}}$, where the sum extends over all sequences $\mathfrak{a}$ for which $\sum n a_n = q$. The set $\mathfrak{G}$ of all formal infinite series $\phi(\mathfrak{d}) = \sum \alpha(\mathfrak{a}) \binom{\mathfrak{d}}{\mathfrak{a}}$ with integral coefficients is, with an obvious definition of addition, a commutative additive group. Furthermore $\mathfrak{G}$ becomes a commutative (associative) ring when 'multiplication' is defined by means of the formula

$$\binom{\mathfrak{d}}{\mathfrak{a}} * \binom{\mathfrak{d}}{\mathfrak{b}} = \binom{\mathfrak{a}+\mathfrak{b}}{\mathfrak{a}} \binom{\mathfrak{d}}{\mathfrak{a}+\mathfrak{b}}.$$

In this ring, form the determinant $\phi^{(\lambda_1,\cdots,\lambda_k)}(\mathfrak{d})$ of the $(k+1)$-rowed matrix whose first row consists entirely of units and whose $(i+1, j+1)$th element, for $i \geq 1$, $j \geq 0$, is $\pi_{\lambda_i - i + j}$. The main result is then as follows: If $(\lambda) : m = \lambda_0 + \lambda_1 + \cdots + \lambda_k$, $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ is a partition of m, and if $\mathfrak{a} = (a_1, a_2, \cdots, a_m, 0, 0, \cdots)$ is a sequence with $\sum_n n a_n = m$, then $\phi^{(\lambda_1,\cdots,\lambda_k)}(\mathfrak{a}) = \chi_{\mathfrak{a}}^{(\lambda)}$, the value of the simple character $\chi^{(\lambda)}$ of $\mathfrak{S}_m$ for the class with a_n cycles of length n $(1 \leq n \leq m)$. The paper also contains explicit formulae giving the functions $\phi^{(\lambda_1,\cdots,\lambda_k)}(\mathfrak{d})$ for all $(\lambda_1, \cdots, \lambda_k)$ such that $\lambda_1 + \cdots + \lambda_k \leq 6$.

H. K. Farahat (Sheffield)

4787:

Hancock, V. Ray. On complete semimodules. Proc. Amer. Math. Soc. **11** (1960), 71–76.

This paper grew out of the discovery of an incorrect statement in a paper by R. Wiegandt [Acta Sci. Math. Szeged **19** (1958), 219–223; MR **20** #6476]. Wiegandt asserted that a semimodule is complete if and only if it is divisible. (For the definitions cf. the above quoted review.) The correct statement is now proved to be: "A semimodule is complete if and only if it is a divisible (abelian) group." Furthermore, divisible semimodules which are not necessarily groups are discussed. In connection with the second theorem of Wiegandt's paper (superfluous after the correction), it is shown that not every divisible semimodule is uniquely expressible as a direct sum in a way suggested by the theory of divisible abelian groups. However, the author indicates conditions for other ways of direct decomposition.

R. Artzy (Chapel Hill, N.C.)

4788:

Saitô, Tôru. Note on left simple semigroups. Proc. Japan Acad. **35** (1959), 427–430.

For every left-simple semigroup S the author obtains a universal 'semigroup with idempotent', i.e., a semigroup T with an idempotent and a homomorphism φ of S onto T such that any homomorphism of S onto any semigroup with idempotent may be factored by φ. The semigroup T is the direct product of a group T_1 and a semigroup T_0 of idempotents satisfying $xy = x$ $(x, y \in T_0)$; here T_1 is the largest homomorphic image group of S [as obtained in Saitô, same Proc. **34** (1958), 664–667; **35** (1959), 114; MR **21** #4192a, b], while T_0 is a certain homomorphic image of the core (loc. cit.) of S.

P. M. Cohn (Manchester)

4789:

Warner, Seth. Mathematical induction in commutative semigroups. Amer. Math. Monthly **67** (1960), 533–537.

Explicit descriptions are given for the commutative semigroups D with composition $+$ which satisfy: (1) there exist $\beta \in D$ such that if a subset S of D contains β and $x+\beta$ for all $x \in S$, then $S = D$; or respectively (2) there exist $\alpha, \beta \in D$ such that if a subset S of D contains α and $x+\beta$ for all $x \in S$, then $S = D$. The positive and the non-negative integers, respectively, furnish examples for (1) and (2).

L. Fuchs (Budapest)

4790:

Sade, A. Multigroupoïde sur un système demosien de groupoïdes. Colloq. Math. **7** (1959/60), 181–185.

The author considers a set Φ of groupoids, all defined on the same set E and interrelated by this condition: for each $\phi_1, \phi_2 \in \Phi$ there is at least one $\phi_3 \in \Phi$ such that for all $x, y, z \in E$, $(x\varphi_1 y)\phi_3 z = x\varphi_3(y\varphi_3 z)$. This defines a multigroupoid (E, Φ) with operation $(\cdot)$ on the set Φ with $\phi_3 \in \phi_1 \cdot \phi_2$.

Relations between the individual groupoids $\phi \in \Phi$ and the multigroupoid (E, Φ) are examined. For example, idempotents, right units, right zeroes in (E, Φ) are related to the presence of semigroups in Φ. Particular attention is given to a Φ consisting of isotopes of a fixed group.

S. Stein (Davis, Calif.)

4791:

Preston, G. B. Congruences on Brandt semigroups. Math. Ann. **139**, 91–94 (1959).

A Brandt semigroup is defined as a semigroup obtained from a Brandt groupoid [H. Brandt, Math. Ann. **96** (1927), 360–366] by adjunction of a zero element and by putting all products not defined in the groupoid equal to zero. Brandt semigroups are known to be completely simple, and their defining matrix is a square identity matrix [A. H. Clifford, Amer. J. Math. **64** (1942), 327–342; MR **4**, 4]. Let I be an arbitrary set, $B = I \times I$, with a partial product in B defined by $(i, j)(j, k) = (i, k)$. Then B is a Brandt groupoid, and any Brandt groupoid is isomorphic to the direct product $G \times B$ for some group G and some I. If the adjunction of a zero is denoted by a zero-subscript, G is called the basic group of the Brandt semigroup $(G \times B)_0$, B_0 its structure semigroup.

Theorem 1: A semigroup is a Brandt semigroup if and only if it is a completely simple inverse semigroup with zero. [Cf. also W. D. Munn, Proc. Cambridge Philos. Soc. **53** (1957), 5–12; MR **18**, 489.] Corollary: A proper (i.e., consisting not merely of a single idempotent) homomorphic image of a Brandt semigroup is itself a Brandt semigroup.

In dealing with congruences on Brandt semigroups the paper uses results about inverse semigroups obtained by the author [J. London Math. Soc. **29** (1954), 396–403; MR **16**, 215]. A congruence is called proper if under it at least two elements of the semigroup are not equivalent. Theorem 2: There is a (1, 1)-correspondence between the proper congruences on a Brandt semigroup and the normal subgroups of its basic group; moreover, if ρ is a proper congruence on a Brandt semigroup S, then S and S/ρ have isomorphic structure semigroups.

R. Artzy (Chapel Hill, N.C.)

4792:

Boccioni, Domenico. Condizioni di associatività negli ipergruppoidi commutativi. Rend. Sem. Mat. Univ. Padova **29** (1959), 215–226.

Un multigroupoïde M est une application de l'ensemble E^2 sur celui des parties de E [Drbohlav, Czechoslovak Math. J. 7 (82) (1957), 183–190; MR 19, 730]; on suppose ici $xy \neq \emptyset$. Si $X \subset E$, $Y \subset E$, le produit XY est défini dans M comme $\bigcup(xy)$, $x \in X$, $y \in Y$. M est propre si tout xy ne se réduit pas à un élément unique de E. Dans deux travaux antérieurs [pour la terminologie, cf. MR 20 #4513, #4514], l'auteur a montré que les conditions d'associativité sur un multigroupoïde de cardinal supérieur à 2 sont indépendantes. Il trouve maintenant ceci: Soit H un multigroupoïde commutatif propre, sur un ensemble E, de cardinal v. Soit Σ l'ensemble des v^3 conditions d'associativité de H. Tous les sous-ensembles $\Sigma' \subseteq \Sigma$ indépendants et équivalents à Σ (ou $\Sigma' \Rightarrow \Sigma$) s'obtiennent de la manière suivante: (i) dans chacun des couples non ordonnés $((x, x, y); (y, x, x))$, $(x \neq y \in E)$ on choisit arbitrairement un triplet; (ii) si v surpasse 2, dans chacun des 6-uples $((x, y, z); (x, z, y); (y, x, z); (y, z, x); (z, x, y); (z, y, x))$ on choisit deux triplets quelconques non opposés, l'opposé de (x, y, z) étant (z, y, x). Ces résultats confirment ceux de G. Szász [Acta Sci. Math. Szeged 15 (1953), 20–28; 15 (1954), 130–142; Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 3 (1953), 569–577; 3 (1954), 97–109; MR 15, 95; 773; 390; MR 16, 441] et Al. Climescu [Bull. Inst. Politehn. Iasi (N.S.) 1 (1955), 1–9; MR 18, 466] dans le cas général, et les complètent pour $v = 2$ et 3. *A. Sade* (Marseille)

4793:

Ginzburg, Abraham. Systèmes multiplicatifs de relations. Boucles quasi associatives. C. R. Acad. Sci. Paris 250 (1960), 1413–1416.

Soit N un ensemble quelconque non vide, M l'ensemble produit NN, Q l'ensemble des parties de M et $\mathscr{R}$ un sous-ensemble de Q, représentant un système de relations binaires sur N ($\mathscr{R} = R, R', R'', \cdots$), ne contenant pas l'ensemble vide, et dont tous les éléments sont disjoints. On désigne par RR' la relation composée de R et R' (ensemble des éléments $(a, c) \in M$ tels que $\exists\, b \in N$, $(a, b) \in R$, $(b, c) \in R'$) et on suppose que $\forall\, R, R' \in \mathscr{R}$, $\exists\, R'' \in \mathscr{R}$, $RR' \subset R''$ $(R, R' \neq \emptyset)$. Cela définit sur $\mathscr{R}$ une loi de composition $R \cdot R' = R''$. Si $\mathscr{R}$ contient, en même temps que toute R, sa symétrique R^- [$(a, b) \in R \Leftrightarrow (b, a) \in R^-$], on notera le système $\mathscr{R}_s$. Si $\mathscr{R}$ est fermé par rapport à $(\cdot)$, on notera $\mathscr{R}_c$. Si $\mathscr{R}$ est une partition de M on désignera le système par $\mathscr{P}$ au lieu de $\mathscr{R}$. Un quasigroupe G avec un élément neutre e est dit avoir la propriété inverse (IP), si $\forall\, a, b \in G$, $aa' = a'a = e \Rightarrow a'(ab) = (ba)a' = b$ [R. Artzy, Proc. Amer. Math. Soc. 6 (1955), 448–453, MR 16, 1083; p. 448]. Alors: (i) Tout système symétrique fermé, $\mathscr{R}_{sc}$, est un quasigroupe IP et réciproquement. (ii) Toute $\mathscr{P}_c$ associative avec un seul élément idempotent est un groupe. (iii) On peut construire, à partir d'un groupe infini quelconque, ou d'un groupe fini d'ordre supérieur à 4, une $\mathscr{P}_{sc}$ contenant un sous-quasigroupe d'indice 2, isomorphe à ce groupe, sans que cette $\mathscr{P}_{sc}$ soit elle-même un groupe. Les quasigroupes qui peuvent être représentés comme une $\mathscr{P}_{sc}$ sont dits quasi-associatifs. (iv) Étant donné un IP quasigroupe possédant un sous-quasigroupe propre, L, il existe une relation associative telle que, si L satisfait à cette condition, alors le quasigroupe donné n'est pas quasi-associatif.

A. Sade (Marseille)

4794:

Bruck, R. H. Normal endomorphisms. Illinois J. Math. 4 (1960), 38–87.

This is the paper whose publication was already announced by the author on p. 72 of his book, *A survey of binary systems* [Springer, Berlin, 1958; MR 20 #76], as an amplification of the theory of normal endomorphisms of loops.

In addition to the normal endomorphisms as defined in his book, several new concepts are introduced, foremost among them are the following two. A "seminormal endomorphism", θ, is defined as an endomorphism satisfying the identities (*) $(x, y)\theta = (x\theta, y) = (x, y\theta)$, $(x, y, z)\theta = (x\theta, y, z) = (x, y\theta, z) = (x, y, z\theta)$, where (a, b) is the commutator of a and b, and (a, b, c) the associator of a, b, c. An endomorphism θ is called "strongly normal" if θ' is an endomorphism and if $\theta\theta'$ maps the loop into its center. Here θ' is defined by $\theta + \theta' =$ the identity mapping. The motivation for exploring these new types of endomorphisms is given by the attempt to find a loop-theoretic generalization of the concept of the normal endomorphism of groups. Now, in a Moufang loop the identities (*) are equivalent to the requirement that θ commute with every inner mapping. Thus the seminormal endomorphism appears as a reasonable generalization of the normal endomorphism of a group as usually defined. Furthermore, an equivalent definition of a normal endomorphism θ of a group requires θ' to be an endomorphism and $\theta\theta'$ to map the group into its center; this suggests the definition of the strongly normal endomorphism of loops.

The investigation of the properties and mutual relations of the different types of "normal" endomorphisms in loops in general and in disassociative loops, Moufang loops and commutative Moufang loops, in particular, is carried out rather thoroughly, and only a few main results can be mentioned here. Every strongly normal endomorphism is normal, every normal endomorphism is seminormal. There exists a commutative Moufang loop for each of the three types: (i) possessing an endomorphism θ which is not seminormal; (ii) possessing a seminormal θ which is not normal; (iii) possessing a normal θ which is not strongly normal. The normal, and therefore also the strongly normal, endomorphisms of a loop commute with all its inner mappings. For a group the class of all seminormal endomorphisms consists of all endomorphisms commuting with all its inner automorphisms, and so does the class of all strongly normal endomorphisms; therefore for a group the usual and the loop-theoretic definitions of a normal endomorphism indeed coincide. Another result: if θ is a normal or strongly normal endomorphism, then also θ' is a normal or strongly normal endomorphism, respectively. For seminormal endomorphisms this last statement is proved in Moufang loops only.

R. Artzy (Chapel Hill, N.C.)

TOPOLOGICAL GROUPS AND LIE THEORY

4795:

Ghika, Alexandru. Groupes à racines topologiques localement paraconvexes. II. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) 2 (50) (1958), 285–306.

As a generalization of locally convex topological vector

spaces, the author introduces locally paraconvex topological groups. A topological group is said to be locally paraconvex, if the group is divisible (but in general non-abelian) and if the topology satisfies a condition similar to the local convexity of topological vector spaces. Semi-norms on a group are defined in a natural way. It is shown that the locally paraconvex topologies on a divisible group are precisely those which can be defined by a family of semi-norms satisfying certain conditions. Definitions of convex sets, absorbant sets, etc. in a divisible group are given in part I [same Bull. **2** (50) (1958), 27–42; MR **22** #2661]. *Ky Fan* (Detroit, Mich.)

4796:

Helmberg, Gilbert. Ein Satz über Gleichverteilung in kompakten Gruppen. II. Monatsh. Math. **63** (1959), 368–377.

The author extends earlier results [Pacific J. Math. **8** (1958), 227–241; MR **21** #2020] about equidistribution in a compact group G, all based in effect on the Peter-Weyl theorem. Sample result: Let $\{H_k\}_1^n$ be a set of subgroups of G; let $\{h_k\}_1^\infty$ be a sequence in H_k, equidistributed; suppose for each irreducible representation R of G there exists k such that the restriction $R|H_k$ does not contain the trivial representation. Then the elements of the form $h_{1\nu_1}h_{2\nu_2}\cdots h_{n\nu_n}$, if properly arranged, form a sequence equidistributed in G; any arrangement compatible with the partial order defined by $\max(\nu_1, \cdots, \nu_n)$ is admitted. The theorems are specialized to finite and to abelian groups. *H. Samelson* (Ann Arbor, Mich.)

4797:

Helmberg, Gilbert. Generating sets of elements in compact groups. Pacific J. Math. **9** (1959), 1083–1096.

The author develops some consequences of the Peter-Weyl theorem. Main result: Let M_1, M_2 be subsets of the compact group G; then M_2 is contained in the (closed) subgroup $H(M_1)$ generated by M_1 if and only if $d^{(\pi)}(M_1 \cup M_2) = d^{(\pi)}(M_1)$ for every representation π, where $d^{(\pi)}(M)$ is the dimension of the space consisting of all vectors in the representation space of π that are fixed under the action of the elements of M. Similarly: If H is a closed subgroup, then $\sum r_\lambda d^{(\lambda)}(H) = 1/\mu(H)$, where the λ are the irreducible representations of G, r_λ is the degree of λ, and $\mu(H)$ is the Haar measure of H in G. The proofs start with the observation that the space of continuous functions on G/H can be identified with the space of continuous functions on G, invariant under right translation by H. *H. Samelson* (Ann Arbor, Mich.)

4798:

Koecher, Max; Roelcke, Walter. Diskontinuierliche und diskrete Gruppen von Isometrien metrischer Räume. Math. Z. **71** (1959), 258–267.

The connection between the discontinuity of a group of homeomorphisms G operating on a space X and the discreteness of such a group in a suitable topology has first been realized by C. L. Siegel [Ann. of Math. (2) **44** (1943), 674–689; MR **5**, 228; lemma 6]. The authors study this relation in the case where X is a metric space and G a group of isometries. Using results of van Dantzig and van der Waerden [Abh. Math. Sem. Univ. Hamburg **6** (1928), 367–376], they easily show that a group of isometries Γ operating on a locally compact, connected metric space X, and discrete in the compact-open topology, is discontinuous on X. They prove a similar result for not necessarily connected X if the topology on Γ is defined somewhat differently. *J. de Groot* (Amsterdam)

4799:

Nagano, Tadashi. On some compact transformation groups on spheres. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **9** (1959), 213–218.

This paper studies the differentiable action of a compact connected Lie group on an homology n-sphere with some orbit $(n-1)$-dimensional. Then it is known that all orbits are $(n-1)$-dimensional except for precisely two orbits F_1 and F_2 of lower dimension, and that all the $(n-1)$-dimensional orbits are of the same type F. The results found here concern the homology of F, F_1, and F_2; there are applications to examples.

D. Montgomery (Princeton, N.J.)

4800:

Seligman, George B. On automorphisms of Lie algebras of classical type. II. Trans. Amer. Math. Soc. **94** (1960), 452–482.

This paper is a continuation of a previous work of the author [same Trans. **92** (1959), 430–448; MR **21** #5695], and we shall use in this review the notations set down in the review of the first paper. The author's first result on algebras of type A is the following theorem. Let $\mathfrak{L}$ be the Lie algebra of n by n matrices of trace zero over an arbitrary field F of characteristic $\neq 2, 3$ whose characteristic does not divide n $(n>1)$. Then every automorphism of $\mathfrak{L}$ is of the form $X \to A^{-1}XA$ or $X \to -A^{-1}X'A$, A non-singular over F. The group of invariant automorphisms is the group of all automorphisms $X \to U^{-1}XU$, where U is unimodular, and is isomorphic to the group PSL(n, F). If F has characteristic $p>11$, and $\mathfrak{R}$ is the Lie algebra of n by n matrices over F where $p|n$, and $\mathfrak{L} = \mathfrak{R}/F \cdot 1$, then every automorphism of $\mathfrak{R}$ is of the form $X \to A^{-1}XA$ or $X \to -A^{-1}X'A$, A non-singular over F, and the automorphisms of $\mathfrak{L}$ are all induced in the obvious way by automorphisms of $\mathfrak{R}$. The group of invariant automorphisms of $\mathfrak{L}$ is again isomorphic to PSL(n, F). The group G' of Chevalley [Tôhoku Math. J. (2) **7** (1955), 14–66; MR **17**, 457] formed over F from a complex simple Lie algebra of type A_{n-1} is isomorphic to PSL(n, F).

For algebras of type C, the author proves that if $\mathfrak{L}$ is the Lie algebra of skew transformations on a vector space V of dimension >2 carrying a nondegenerate alternate bilinear form (x, y) over a field F of characteristic $p>7$ or $p=0$, then every automorphism is of the form $X \to A^{-1}XA$, where A satisfies $(xA, yA) = (x, y)\alpha$, $x, y \in V$, for some fixed $\alpha \neq 0$ in F. The group of invariant automorphisms, as well as the Chevalley group G', is isomorphic to PSp(V).

For algebras of type B, the author proves that if $\mathfrak{L}$ is the Lie algebra of skew transformations of a vector space V of odd dimension $2n+1 \geqq 3$, carrying a non-degenerate symmetric bilinear form of maximal index and satisfying the discriminant condition $(-1)^n\Delta \in F^2$, where F is of characteristic $\neq 2, 3, 5, 7$, then every automorphism of $\mathfrak{L}$ is of the form $X \to A^{-1}XA$, where A satisfies $(xA, yA) = (x, y)\alpha, x, y \in V$, for fixed $\alpha \neq 0$ in F. The group of invariant

automorphisms is isomorphic to the commutator group $\Omega(V)$ of the orthogonal group on V, and is isomorphic with the Chevalley group G'. For a Lie algebra of type D acting on a space of dimension $2n$, $n \neq 4$, the automorphisms are as for type B, while if $n=4$ some other possibilities caused by the principle of triality arise. In all cases the group of invariant automorphisms of a Lie algebra of type D is isomorphic with $P\Omega(V)$, and is also isomorphic to the group G' of Chevalley.

Finally, let $\mathfrak{C}$ be the split Cayley algebra over a field F of characteristic $\neq 2, 3$, and let $\mathfrak{L}$ be the derivation algebra of $\mathfrak{C}$. Then $\mathfrak{L}$ is of type G_2. If the characteristic of F is not 5, 7, 11, 13, 17, then the group of invariant automorphisms is the full automorphism group of $\mathfrak{L}$, and is isomorphic to the group G' of Chevalley.

C. W. Curtis (Madison, Wis.)

MISCELLANEOUS TOPOLOGICAL ALGEBRA

4801:

Swingle, Paul M. The existence of widely connected and biconnected semigroups. Proc. Amer. Math. Soc. **11** (1960), 243–248.

A connected set W is widely connected if each non-degenerate connected subset of W is dense in W. A connected set B is biconnected if it is not the union of two disjoint non-degenerate connected sets; a point $d \in B$ is a dispersion point of B if $B-d$ is totally disconnected. The author, making a certain algebraic assumption (AA), constructs two examples: (1) a widely connected metric topological group; in particular, this is a connected topological group which does not contain an arc; (2) a biconnected topological semigroup with unit and zero, zero being a dispersion point. These examples are constructed in a compact metric indecomposable continuum I which is a topological group, and the algebraic assumption (AA) is as follows: Let g and G each be a product of a finite number of elements of I; let $H(p, G)$ be a similar product including n occurrences of $p \in I$ and n' occurrences of the inverse of p; then we assume that there exist only a countable number of solutions p of $H(p, G)=g$.

R. J. Koch (Baton Rouge, La.)

4802:

Swingle, Paul M. Widely connected and biconnected semigroups. Proc. Amer. Math. Soc. **11** (1960), 249–254.

Terminology is as in the preceding review. Let S be a connected topological semigroup, with multiplication extendable to $\bar{S}$, where $\bar{S}$ is compact metrizable. This note consists of remarks on the structure of S if S is widely connected, or if S is biconnected with dispersion point. The analysis is made in terms of the sets $A=\{a : aS$ is a point, $a \in S\}$, $B=\{b : Sb$ is a point, $b \in S\}$, $C=\{c : c\bar{S} \supset S$, $c \in S\}$, and $D=\{d : \bar{S}d \supset S, d \in S\}$. A very special case of the rather lengthy Theorem 1 is that if S is widely connected, then $S \subset A \cup B \cup C \cup D$. If S is biconnected with dispersion point d, it is shown, for example, that for every $s \in S$, s^2 is a zero for S, or d is a left or right zero or zero.

R. J. Koch (Baton Rouge, La.)

FUNCTIONS OF REAL VARIABLES

See also 4905a–c.

4803:

Pisanelli, Domingos. Sur l'intégration des fonctions rationnelles par des différences subdivisées. Boll. Un. Mat. Ital. (3) **14** (1959), 139–141.

The integral $\int_a^b P(x)/Q(x)\,dx$, where $P(x)$ and $Q(x)$ are polynomials, $Q(x)=x^n+\cdots$, $\deg P<n$, is expressed as the nth divided difference of the function $P(x)\log(b-x) \times (a-x)^{-1}$ with respect to the roots of $Q(x)$.

P. Henrici (Los Angeles, Calif.)

4804:

Usai, Giuseppe. Sui cilindri iscritti in un cono e sui coni circoscritti ad un cilindro. Archimede **11** (1959), 159–163.

A circular right cone of altitude h and radius r has an inscribed cylinder of maximum volume (altitude $h/3$) and an inscribed cylinder of maximum lateral area (altitude $h/2$). In general (for any r and h) no maximum exists for the total surface of an inscribed cylinder, unless $r<h/2$. But there exists a cone of minimum volume, circumscribed to a given cylinder.

S. R. Struik (Cambridge, Mass.)

4805:

Usai, Giuseppe. Sui cilindri iscritti in una sfera. Archimede **11** (1959), 295–301.

With the diameter of a sphere (of radius r) as common axis, an infinite number of right circular cylinders is inscribed in the sphere. The equilateral cylinder, inscribed in the sphere, has maximum lateral surface ($S=2\pi r^2$); the radius of its base $x=r/\sqrt{2}$. It shares its total surface $T=3\pi r^2$ with the cylinder for which $x=3r/\sqrt{10}$. The total surface T has a maximum, $T=\pi r^2(1+\sqrt{5})$. When we integrate S, expressed as function of x, from 0 to r, we obtain the volume of the sphere.

S. R. Struik (Cambridge, Mass.)

4806:

Ganguli, P. L. Construction of a class of steadily increasing continuous functions which are not absolutely continuous. Ann. Univ. Ferrara. Sez. VII (N.S.) **8** (1958/59), 21–27. (Italian summary)

Author's summary: "Starting with a Gilman's non-dense perfect set of measure zero and three convergent series of positive terms satisfying certain prescribed conditions, a class of steadily increasing continuous functions have been constructed which are not absolutely continuous."

T. A. Botts (Charlottesville, Va.)

4807:

Hebroni, Pessach. On the inequality of Schwarz. Riveon Lematematika **13** (1959), 42–45. (Hebrew)

This is a didactic paper in which the author discusses the classical Schwarz inequality as a special case of the corresponding inequality for definite quadratic functionals. Some elementary applications are considered.

M. Schiffer (Stanford, Calif.)

4808:

Kaščenko, Yu. D. On an inequality for differentiable functions of many variables. Uspehi Mat. Nauk **15** (1960), no. 1 (91), 203–206. (Russian)

The author considers a function $f(x)=f(x_1, \cdots, x_n)$ defined on a bounded open set G, in Euclidean n-dimensional space R_n, and such that its partial (generalized) derivatives $f^{(\kappa)}=\partial^k f/\partial x_1^{\kappa_1}\cdots\partial x_n^{\kappa_n}$, where $k=|\kappa|=\kappa_1+\cdots+\kappa_n$, belong to the Lebesgue class $L_p(G)$ whenever $k\leq l$. Let the norm in this class of any function ϕ be denoted by $\|\phi\|$. The author shows that then

$$(*) \qquad \|f^{(\kappa)}\| \leq A\|f\| + B\sum_{|\lambda|=l}\|f^{(\lambda)}\|$$

with $|\kappa|=0, 1, \cdots, l$, the constants A and B being independent of f, provided that for some positive η the part of G at a distance not greater than η from the boundary of G can be covered by the union of a finite set of "elementary" domains G_q ($q=1, \cdots, N$). A domain G_q is called elementary if (i) it is contained in some cylinder C of bounded cross-section, (ii) it contains some finite segment of C, and (iii) the intersection of G_q and any line parallel to the generators of C either is a finite segment of the line or is empty. In particular, G may be a star domain. The author notes that inequalities similar to (*) have been given by, for instance, S. G. Mihlin [*Problema minimuma kvadratičnogo funkcionala*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1952; MR **16**, 41]. In some special cases there is an overlap between the author's results and the results (more elaborate in form) announced more recently by V. P. Il'in [Dokl. Akad. Nauk SSSR **129** (1959), 1214–1217; MR **22** #2882].

H. P. Mulholland (Exeter)

4809:

Gårding, Lars. An inequality for hyperbolic polynomials. J. Math. Mech. **8** (1959), 957–965.

Let the real polynomial P in n-vector arguments $x=\{x_k\}_1^n$ be homogeneous of degree $m\geq 1$. For the real n-vector $a=\{a_k\}_1^n\neq 0$, to say P is 'hyperbolic a' means that for any real x the polynomial $P(sa+x)$ in the single variable s has all m zeros real. Assume this, with $P(a)>0$, and let $C(a)$ denote the cone of all x such that $t\geq 0$ implies $P(ta+x)>0$. It is proved that $C(a)$ is convex, that P is hyperbolic b for any $b\in C(a)$, and that $C(b)=C(a)$. The main result is the inequality $M(x^1, \cdots, x^m)^m \geq P(x^1)\cdots P(x^m)$, where M is the completely polarized form of P and each vector x^i is in $C(a)$; precise conditions for equality are given. Corollary: the set $\{x: P(x)=c, x\in C(a)\}$ is convex for $c>0$ (for $c=0$ it is $C(a)$).

The author states that his general inequality, "which is used in the preceding paper by S. S. Chern [i.e., #4997 below], is an immediate consequence of various properties of hyperbolic polynomials proved in the second chapter of Gårding, Acta Math. **85** (1951), 1–62 [MR **12**, 831], but we shall give an independent proof of it. Since most relevant properties of hyperbolic polynomials enter into this proof, this article contains, in fact, a rather complete theory of homogeneous hyperbolic polynomials."

C. Davis (Providence, R.I.)

4810:

Hriptun, V. G. Functions analytical with respect to a hyperbolic operator involving two independent variables. Dokl. Akad. Nauk SSSR **122** (1958), 26–28. (Russian)

This paper contains extensions, to functions of two variables, of certain results of M. K. Fage [same Dokl. **112** (1957), 1008–1011; MR **20** #1011a] for operator-analytic functions of a single variable. Consider the hyperbolic operator

$$H=\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+p(x, y)\frac{\partial}{\partial x}+q(x, y)\frac{\partial}{\partial y}+r(x, y)$$

on the square D defined by $|x|+|y|<1$; the functions p and q are supposed to be once continuously differentiable in D, and r is continuous in D. A function $f(x, y)$ is called infinitely H differentiable in D provided that $H^k f(x, y)$ is twice continuously differentiable in D, for each $k=0, 1, 2, \cdots$. An infinitely H differentiable function $f(x, y)$ is said to be (H, x) analytic in D provided that for each closed subset R in D there is a positive number $C=C(f, H, R)$ such that

$$\left|\frac{\partial}{\partial x^s}H^k f(x, y)\right| \leq C^{2k+s}(2k+s)!,$$

for $s=0, 1$; $k=0, 1, 2, \cdots$, except possibly for $k=s=0$.

J. B. Diaz (College Park, Md.)

MEASURE AND INTEGRATION

4811:

Nikodým, Otton Martin. Summation of quasi-vectors on Boolean tribes and its application to quantum theories. I. Mathematically precise theory of P. A. M. Dirac's delta function. Rend. Sem. Mat. Univ. Padova **29** (1959), 1–214.

4812:

Chiffi, Antonio. Sulla continuità degli integrali curvilinei. Rend. Sem. Mat. Univ. Padova **29** (1959), 411–430.

L. Tonelli [*Fondamenti di calcolo delle variazioni, Vol. I*, Zanichelli, Bologna, 1921; pp. 293–294] has shown that if $F(x, y)$ and $G(x, y)$ are continuous on a closed and bounded set H of the XY-plane, then the curvilinear integral $I(F, G; \mathfrak{C})=\int_{\mathfrak{C}}(F dx+G dy)$ is continuous according to a specified topology [ibid., p. 72] on every class $\mathfrak{F}$ of curves $\mathfrak{C}$ which are continuous and rectifiable with uniformly bounded length. Here the conditions on F and G are weakened to the assumption that they are quasi-continuous in the sense that $\varepsilon>0$ implies the existence of an open set A with $\mu_e(A)<\varepsilon$, such that F and G are continuous on $H-A$, $\mu_e(E)$ being $\overline{\text{meas}}\, E_x+\overline{\text{meas}}\, E_y$, where E_x and E_y are the projections of E on the X and Y axes, respectively. In addition the class $\mathfrak{F}$ of curves is limited to continuous rectifiable curves contained in H for which the indefinite integrals of the associated Banach function $N(\mathfrak{C}; t)$, which for every t is equal to the number of times the lines $x=t$ and $y=t$ cut $\mathfrak{C}$ [Banach, Fund. Math. **7** (1925), 225–236], are equi-absolutely continuous. The topology on $\mathfrak{F}$ is the same as Tonelli's. In particular, if a sequence of curves $\mathfrak{C}_n$ converges in length to a curve $\mathfrak{C}$ in $\mathfrak{F}$, then $\lim_n I(F, G; \mathfrak{C}_n)=I(F, G; \mathfrak{C})$.

T. H. Hildebrandt (Providence, R.I.)

4813:

Leifman, L. Ya. Remarks on "On conditions for existence of a Kolmogoroff integral and the concept of differential equivalence". Uspehi Mat. Nauk **15** (1960), no. 1 (91), 259–261. (Russian)

A condition for the existence of Kolmogoroff integrals given by the author [Uspehi Mat. Nauk **12** (1957), no. 3 (75), 343–352; MR **20** #952] is generalised to measures with infinite values. If $\varphi(E)$ is a completely additive non-negative set function on subsets of E_0, $f(x)$ defined on E_0, then for the existence of the integral $\int f(x)\varphi(dE_0)$ it is necessary and sufficient that $\int(\omega_f\varphi)(dE_0)=0$ and that, for some partition θ_0E_0, $(R|\overline{f}_x|\varphi)(\theta_0E_0)<\infty$.

J. L. B. Cooper (Cardiff)

4814:

von Krbek. Ausführungen zur Ergodentheorie. Collect. Math. **11** (1959), 49–59.

Expository paper. *E. Nelson* (Princeton, N.J.)

4815:

Abramov, L. M. The entropy of a derived automorphism. Dokl. Akad. Nauk SSSR **128** (1959), 647–650. (Russian)

Let M be a Lebesgue space with measure μ [cf. Rohlin, Uspehi Mat. Nauk **4** (1949), no. 2 (30), 57–128; MR **11**, 40] and let T be an automorphism of M. Let $X\subset M$ be a measurable subset of M such that $\bigcup_{n=0}^{\infty}T^nX=M$; then T induces an automorphism T' of X regarded as a Lebesgue space with measure μ_X ($\mu_X(A)=\mu(A)/\mu(X)$). T' is defined by $T'x=T^kx$, $x\in X$, where k is the smallest positive integer for which $T^kx\in X$ [cf. Kakutani, Proc. Imp. Acad. Tokyo **19** (1943), 635–641; MR **7**, 255]. The main result of this note is the following: let $h(T)$ and $h(T')$ be the entropy of T and T' respectively [cf. Sinaĭ, same Dokl. **124** (1959), 768–771; MR **21** #2036a]. Then $h(T)=h(T')\mu(X)$.

The above theorem is connected with another theorem which states, essentially, that the entropy is distributed 'uniformly' in M. More precisely, let ξ be a decomposition of M into an at most denumerable number of measurable sets C_k. Let $\xi_{T^n}=\bigvee_{i=0}^{n-1}T^i\xi$ denote the common refinement of $\xi, T\xi, \cdots, T^{n-1}\xi$. Let $H(\xi\wedge X)=-\sum_k\mu(C_k\wedge X)\times\log\mu(C_k\wedge X)$. Let $h(T,\xi,X)=\lim_{n\to\infty}n^{-1}H(\xi_{T^n}\wedge X)$. The author proves that this limit always exists provided ξ is suitably and naturally restricted (relative to X) to a certain class Z_X, and that if $h(T,X)=\operatorname{Sup}_{\xi\in Z_X}h(T,\xi,X)$, then $h(T,X)=h(T)\mu(X)$.

The proofs depend on Birkhoff's ergodic theorem and McMillan's theorem. *Y. N. Dowker* (London)

4816:

Abramov, L. M. On the entropy of a flow. Dokl. Akad. Nauk SSSR **128** (1959), 873–875. (Russian)

Let $\{S_t\}$ be a measurable flow acting on a Lebesgue space. It has been proved [Sinaĭ, same Dokl. **124** (1959), 768–771, 1200–1202; MR **21** #2036a, b] that if t_2/t_1 is a rational number then $h(S_{t_2})=|t_2/t_1|h(S_{t_1})$ where $h(S_{t_i})$ is the entropy of the automorphism S_{t_i} ($i=1,2$). The author proves that the above relation holds for all ratios t_2/t_1, i.e., $h(S_t)=|t|h(S_1)$ for $-\infty<t<\infty$. Thus one can define $h(S_t)$ of $\{S_t\}$, satisfactorily, as $h(S_1)$ (compare this definition with the one previously given by Sinaĭ [above reference]).

The author first proves that if $\{T_t\}$ is a flow built on an automorphism T of a Lebesgue space (Y,ν) and under a function $f(y)>0$ with $\int_Y f(y)d\nu=1$ [cf. Ambrose, Ann. of Math. (2) **42** (1941), 723–739; MR **3**, 52], then $h(T_t)=|t|h(T)\nu(Y)$, $-\infty<t<\infty$. The proof of this preliminary result depends on the results of his previous paper reviewed above. This result is then combined with the theorem on the decomposition of measurable flows into an invariant component and an at most denumerable number of components each of which is isomorphic (mod 0) to a flow built under a function [cf. Ambrose and Kakutani, Duke Math. J. **9** (1942), 25–42; MR **3**, 210] to obtain the asserted theorem for any measurable flow.

Y. N. Dowker (London)

FUNCTIONS OF COMPLEX VARIABLES

See also 4890, 4894, 4918, B5247, B5248.

4817:

Knopp, Konrad. ★Elemente der Funktionentheorie. 5te Aufl. Sammlung Göschen Bd. 1109. Walter de Gruyter and Co., Berlin, 1959. 144 pp. DM 3.60.

Contents: I: Complex numbers and their geometric representation. II: Linear functions. III: Sets, sequences and power series. IV: Analytic functions and conformal mapping. V: Elementary functions.—An English translation [Dover, New York, 1953] is listed in MR **14**, 458.

4818:

Miller, Kenneth S. ★Advanced complex calculus. Harper & Brothers, New York, 1959. viii+240 pp. $5.75.

In these days the theory of functions of a complex variable has lost its widespread interest as a field of research. Nevertheless, a knowledge of some parts of the theory is necessary in applied work, differential equations and the integral transform theory for example. This combination of circumstances encourages a hurried and superficial development of only those parts of the theory which are essential to the work in hand. As a consequence students fail to get a clear understanding of the theory or of its contribution to the main stream of mathematics. The book under review is a welcome reversal of this tendency. It is written in the spirit of the author's *Advanced real calculus* [Harper, New York, 1957; MR **19**, 253] (ARC), to which many references are made.

The book starts with the definition of complex numbers as the set of ordered pairs of real numbers, relates this to the representation $a+ib$ and goes on to develop the rules of operation and properties of the complex numbers. Chapter 2 is entitled Topological preliminaries. It introduces the language of sets and neighbourhoods, states the Heine-Borel theorem and refers to (ARC) for its proof. Continuous curves and rectifiability are considered which lead to the concept of functions of bounded variation. It is shown that a function with a bounded derivative is of bounded variation. If an arc of a curve C is continuous and without multiple points it is a simple arc. If it is rectifiable it is a Jordan arc. If it is closed it is a Jordan curve. The chapter closes with a detailed discussion of the Jordan curve theorem. There is a reference to Newman or Dienes for the proof.

Chapter 3 begins with a consideration of limits and continuity of functions and goes on to define the elementary functions, which are introduced by their (real)

series expansion and then extended to the complex plane. Chapter 4 deals with contour integration and the contours are the simple curves defined in chapter 2. The integral of a function along such a curve C is defined and shown to exist for continuous functions. The next step is the proof of the Cauchy integral theorem. The Cauchy integral formula is then proved and used to obtain the usual formulas for derivatives of all orders of analytic functions and to evaluate improper integrals. The chapter ends with the proof of Morera's theorem.

Chapter 5 is concerned with sequences and series of complex numbers and of functions of the complex variable z. Most of the theorems are counterparts of those in (ARC). They are stated precisely in terms of complex numbers and variables, and most of the proofs are given in detail. The Taylor's and Laurent's expansions are included. The chapter ends with the classification of singularities of functions which are analytic except at isolated points. In chapter 7 some of the deeper theorems are given. The fundamental theorem of algebra; in the extended plane a rational function has as many zeros as poles; if $f(z)$ is analytic on an open region and vanishes at a set of points which has a limit point then it vanishes identically; the maximum modulus theorem; Rouché's theorem. The chapter ends with a study of the concept of analytic continuation. Chapter 8 deals with conformal mapping and chapter 9 with the method of the Laplace integral and its application to the solution of linear differential equations. The work on conformal mapping includes a detailed study of the elliptic integral

$$\int_0^z ((1-z^2)(1-k^2z^2))^{-1/2}\,dz.$$

Considering the book as a whole, it is a detailed study of the elements of the complex variable theory which will be a valuable aid to students and instructors in introductory courses. There are exercises at the end of each chapter with a substantial number of problems well chosen to deepen the insight of the reader in the theory of the chapter. The diagrams are well drawn and cleverly designed to illustrate the points under study.

R. L. Jeffery (Kingston, Ont.)

4819:

Sansone, Giovanni; Gerretsen, Johan. ★Lectures on the theory of functions of a complex variable. I. Holomorphic functions. P. Noordhoff, Groningen, 1960. xii+488 pp. Dfl. 45.00, $12.00; cloth: Dfl. 48.75, $13.00.

This is the first volume of a text based on the two-volume work, in Italian, published by the first author in 1947 [MR 8, 143]. The subjects considered in the book are largely of a formal nature. The more geometric aspects of holomorphic functions (including such topics as analytic continuation, Riemann surfaces, conformal mapping) are not taken up in this volume at all, but will presumably be treated in a proposed second part. By assuming knowledge of elementary analysis, by not devoting any space to the axiomatic foundations of the subject, and by omitting details of proofs, the authors have managed to present a large number of theorems and formulas, going well beyond what is to be found even in the more voluminous of recent texts on complex variables. Many types of expansions for holomorphic functions are given, and special functions and classes of functions are discussed at some length.

Some idea of the material covered is conveyed by the following list of chapter headings: 1. Holomorphic functions. Power series as holomorphic functions. Elementary functions. 2. Cauchy's integral theorem and its corollaries. Expansion in Taylor series. 3. Regular and singular points. Residues. Zeros. 4. Weierstrass's factorization of integral functions. Cauchy's expansion in partial fractions. Mittag-Leffler's problem. 5. Elliptic functions. 6. Integral functions of finite order. 7. Dirichlet series. The zeta function of Riemann. The Laplace integral. 8. Summability of power series outside the circle of convergence. Sum formulas. Asymptotic series.

The book contains no exercises. However a good many illustrative examples are given.

W. J. Thron (Boulder, Colo.)

4820:

★Analytic functions. Princeton Mathematical Series, No. 24. Princeton Univ. Press, Princeton, N.J., 1960. vii+197 pp. $5.00.

From the preface: "This book contains the principal addresses delivered at the Conference on Analytic Functions held September 2–14, 1957 at the Institute for Advanced Study, Princeton." The papers—by L. V. Ahlfors, M. Behnke and H. Grauert jointly, L. Bers, M. Heins, J. A. Jenkins, K. Kodaira, R. Nevanlinna, and D. C. Spencer—will be given separate reviews.

4821:

Specht, Wilhelm. Eine Bemerkung zum Satze von Gauss-Lucas. Jber. Deutsch. Math. Verein. **62**, Abt. 1, 85–92 (1959).

Denote by $\mathfrak{P}(f)$ the smallest convex region in the complex plane containing all the roots $\zeta_1, \zeta_2, \cdots, \zeta_n$ of a polynomial $f(z)=a_0+a_1z+\cdots+a_nz^n$, and by $\mathfrak{P}^*(f)$ the smallest convex region containing the n^2-n numbers $\eta_{\kappa\lambda}=[(n-1)\zeta_\kappa+\zeta_\lambda]/n$, $\kappa\neq\lambda$, $1\leq\kappa,\lambda\leq n$. The author proves two theorems which are generalizations of the well-known Gauss-Lucas theorem: (1) The region $\mathfrak{P}^*(f)$ of a polynomial $f(z)$ contains the region $\mathfrak{P}(f')$ of its derivative. (2) If $\zeta_1, \zeta_2, \cdots, \zeta_n$ are distinct complex numbers and $p_1, p_2, \cdots, p_n$ arbitrary positive numbers with $p_1+p_2+\cdots+p_n=p$, denote by $\mathfrak{P}(\zeta, p)$ the smallest convex region enclosing the n^2-n numbers $\zeta_{\kappa\lambda}=[(p-p_\kappa)\zeta_\kappa+p_\kappa\zeta_\lambda]/p$, $\kappa\neq\lambda$, $1\leq\kappa,\lambda\leq n$. Then all the zeros of the rational function $F(z)=\sum_\nu p_\nu/(z-\zeta_\nu)$, $1\leq\nu\leq n$, are contained in $\mathfrak{P}(\zeta, p)$.

J. W. Andrushkiw (Newark, N.J.)

4822:

Parodi, Maurice. Sur la détermination d'une borne supérieure des zéros d'un polynome. C. R. Acad. Sci. Paris **250** (1960), 1953–1954.

By Rouché's theorem the author proves that if $M=\max(|a_k|/\lambda_k)^{1/k}$ $(k=1, 2, \cdots, n)$ for given positive reals λ_k, then all the zeros of the polynomial $f(z)=z^n+a_1z^{n-1}+\cdots+a_n$ lie in the disk $|z|\leq M$. {Reviewer's note: This is a known result due to Fujiwara [Tôhoku Math. J. **9** (1916), 167–171].}

M. Marden (Milwaukee, Wis.)

4823:

Vincze, István. On the theorem of Eneström-Kakeya. Mat. Lapok **10** (1959), 127–132. (Hungarian. Russian and English summaries)

If for the complex numbers $\alpha_0, \alpha_1, \cdots, \alpha_n$ there is a pair (φ, ψ) of angles with $|\varphi - \psi| < \pi$, $\alpha_k = a_k e^{i\varphi} + b_k e^{i\psi}$ $(k = 0, 1, \cdots, n)$ and $a_0 \geq a_1 \geq \cdots \geq a_n \geq 0$, $b_0 \geq b_1 \geq \cdots \geq b_n \geq 0$, then the equation $\alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n = 0$ has no roots inside the circle with radius $\rho_0 = |\alpha_0|/(a_0 + b_0) \geq |\cos(\frac{1}{2}\varphi - \frac{1}{2}\psi)|$. This theorem is an analogue, for complex coefficients, of the theorem of Eneström-Kakeya.

A. Kertész (Debrecen)

4824:

Melzak, Z. A. Power series representing certain rational functions. Canad. J. Math. **12** (1960), 20–26.

A complex number $\alpha \neq 0$ is said to be pseudo-rational if $\alpha/|\alpha|$ is a root of unity. Let $\mathfrak{R}$ be the set of all rational functions $f(z) = \sum_0^\infty f_n z^n$ regular at $z = 0$, and let $\mathfrak{P}$ be the subset of those $f(z)$ that have real coefficients f_n and possess only pseudo-rational poles. Then with $f(z)$ also $\chi_f(z) = \sum_0^\infty \operatorname{sgn} f_n \cdot z^n$ belongs to $\mathfrak{P}$. Further $\mathfrak{P}$ is a subalgebra over the real numbers of $\mathfrak{R}$ if the sum of two elements $f(z)$ and $g(z) = \sum_0^\infty g_n z^n$ is defined as usual and the product is defined by $f \circ g(z) = \sum_0^\infty f_n g_n z^n$. Further $\mathfrak{P}$ is a lattice under the operations $f \cup g(z) = \sum_0^\infty \max(f_n, g_n) z^n$, $f \cap g(z) = \sum_0^\infty \min(f_n, g_n) z^n$.

K. Mahler (Manchester)

4825:

Ionescu Tulcea, Alexandra. Analytic continuation of random series. J. Math. Mech. **9** (1960), 399–410.

The author has obtained some very interesting results about the analytic aspects of the ergodic theory of random series and some related properties of unitary operators. Let (X, B, μ) be a non-atomic measure space of finite total measure and let τ denote an ergodic transformation on X. For fixed $f \in L^p(X, B, \mu)$, $1 \leq p \leq \infty$, the author considers the power series $F(f; x, z) = \sum_0^\infty f(\tau^n(x)) z^n$, $x \in X$, $|z| < 1$. If $f \neq 0$, it is shown that for almost all x the radius of convergence of the series $F(f; x, z)$ is one. The main result states that the set of functions $f \in L^p(X, B, \mu)$ such that the unit circle $[z; |z| = 1]$ is the natural boundary of the power series $F(f; x, z)$ for almost all $x \in X$, is a set of the second category in $L^p(X, B, \mu)$, namely an everywhere dense G_δ. The proof of this result is very ingenious. The second part of the paper is concerned with the set of unitary operators $\mathfrak{U}$ on an infinite dimensional Hilbert space H. It is shown that for an operator $U \in \mathfrak{U}$ the following three assertions are equivalent. (i) The conjugate class of U with respect to $\mathfrak{U}$, i.e., the set $[V^{-1}UV; V \in \mathfrak{U}]$, is dense in $\mathfrak{U}$ in the strong operator topology. (ii) The spectrum of U coincides with the whole unit circle. (iii) The set of all elements $f \in H$ such that the unit circle is the natural boundary of the vector-valued analytic function $\sum_0^\infty U^{n+1} f z^n$ is a set of the second category in H.

R. S. Phillips (Stanford, Calif.)

4826:

Grunsky, Helmut. Eine Grundaufgabe der Uniformisierungstheorie als Extremalproblem. Math. Ann. **139**, 204–216 (1960).

Extremal methods are used to solve the following problem. Let $w = f(z)$ be a holomorphic function on the perimeter of the unit circle $|z| < 1$ which maps it univalently onto the perimeter of the infinite region $|w| > 1$. This identification of the boundaries of $|z| < 1$ and $|w| > 1$ generates a closed Riemann surface. Schlicht functions are sought which transform that surface onto the entire complex plane.

P. R. Garabedian (New York)

4827:

Curtis, H. B., Jr. Some properties of functions which uniformize a class of simply connected Riemann surfaces. Proc. Amer. Math. Soc. **10** (1959), 525–530.

Es sei W die einfach zusammenhängende Riemannsche Fläche, die nur über $-a, a, b, \infty$ verzweigt ist $(0 < a < b)$ und, bis auf zwei schlichte Punkte über $-a, \infty$, nur genau zweiblättrige Windungspunkte über $-a, a, b, \infty$ hat [vgl. Ullrich, Nachr. Ges. Wiss. Göttingen I, N.F. **1** (1936), 135–150; Teichmüller, Deutsche Math. **7** (1944), 360–368; MR **8**, 327; Abb. 1]; es sei $\mathscr{S}$ die Klasse Riemannscher Flächen, die aus W dadurch entstehen, dass die Windungspunkte über $\pm a$ nach $(-1)^{n+1} a_n$, die Windungspunkte über b nach b_{2n-1} verschoben werden $(a_n > 0; a_{2n-1} < b_{2n-1}; n = 1, 2, \cdots)$. Verf. befasst sich mit Eigenschaften der Funktionen $f(z)$, die eine Fläche $S \in \mathscr{S}$ erzeugen. Eine Methode, die von MacLane [Trans. Amer. Math. Soc. **62** (1947), 99–113; MR **9**, 85] und H. E. Taylor [Proc. Amer. Math. Soc. **4** (1953), 52–68; MR **14**, 744] bei einer ähnlichen Flächenklasse benutzt wurde, wird verwendet, um $f(z) = e^{\delta z} P(z)$ nach geeigneter Normalisierung zu gewinnen $(\delta \leq 0$, $P(z)$ kanonisches Produkt vom Geschlecht Null mit doppelten negativen Polstellen und einfachen positiven Nullstellen), und um umgekehrt zu zeigen, dass Funktionen $f(z)$ dieser Struktur Flächen $S \in \mathscr{S}$ erzeugen. {Die Durchführung enthält häufig Ungenauigkeiten.} $\delta < 0$ impliziert $a_n = O(e^{-p n})$ und $b_{2n-1} \geq A e^{p n}$ mit $p > 0$, $A > 0$. {Der gegebene Beweis ist lückenhaft.}

F. Huckemann (Giessen)

4828:

Constantinescu, Corneliu; Cornea, Aurel. Über einige Probleme von M. Heins. Rev. Math. Pures Appl. **4** (1959), 277–281.

Definitive answers are given to two questions raised by the reviewer in Ann. of Math. (2) **55** (1952), 296–317 [MR **13**, 643]. The authors show: (1) There exists a parabolic Riemann surface with a single Kerékjártó boundary element which admits a meromorphic function having an assigned closed set of zero capacity in the extended plane as its Picard exceptional set. (2) There exists a parabolic Riemann surface with a single Kerékjártó boundary element with ends having harmonic dimension $2^{\aleph_0}$ (i.e., admitting $2^{\aleph_0}$ mutually non-proportional minimal positive harmonic functions which vanish continuously on the relative boundary). The following forms an essential part of the argument: Given a non-compact parabolic Riemann surface G, there exists a parabolic Riemann surface F with a single Kerékjártó boundary element which admits a conformal map f (of infinite valence) onto G having certain ramification and covering properties. The latter theorem is established by construction from suitably slit examplars of G.

M. H. Heins (Urbana, Ill.)

4829:

Mel'nik, I. M. Branch point indices of a multiple-valued function. Dokl. Akad. Nauk SSSR **134** (1960), 521–524 (Russian); translated as Soviet Math. Dokl. **1**, 1102–1104.

An index theorem of the Morse type is established for

certain special classes of multiply-valued analytic functions that have domains ramified over a finite number of points and admit local representations in the neighborhood of these points of a certain prescribed nature. For results of this kind, cf. Povolockiĭ, same Dokl. **129** (1959), 265–267 [MR **22** #757]. A detailed description cannot be given briefly. *M. H. Heins* (Urbana, Ill.)

4830:

Roelcke, Walter. Eine Identität für Integrale über Fundamentalbereiche von Grenzkreisgruppen. Math. Ann. **140** (1960), 408–413.

Let Γ be a discrete sub-group of the real unimodular group in $\mathfrak{H}$ the upper-half $(\tau=x+iy)$ plane. The fundamental domain $\mathfrak{F}$ has finite content $\int d\omega=f$, where $d\omega=dxdy/y^2$, and the transformations of Γ with fixed point at ∞ are of the form U^n, where U is $\{\tau'=\tau+1\}$. Let $\mathfrak{N}_\eta$ be the region $\{0\leq x<1 \text{ and } y\geq\eta\}$, and $g(\tau)$ be a complex Lebesgue-measurable function defined on $\mathfrak{F}$ and extended to $\mathfrak{N}_\eta$. Then

$$\int_{\mathfrak{F}} g\,d\omega = F\lim_{\eta\to 0}\eta\int_{\mathfrak{N}_\eta} g\,d\omega$$

with the left-hand integral existing only if the right-hand limit exists. To prove this, one first transforms the various pieces of $\mathfrak{N}_\eta$ into $\mathfrak{F}$ obtaining a "tally" v_η so that $\int_{\mathfrak{N}_\eta} g\,d\omega=\frac{1}{2}\int_{\mathfrak{F}} gv_\eta\,d\omega$ (the factor $\frac{1}{2}$ arising from the equivalence of matrices S and $-S$ corresponding to a given transformation). Letting $\mathfrak{S}$ denote a minimal set of S for which $\{U^nS\}=\Gamma$, then $v_\eta(\tau)=\sum 1$ over $\{S\in\mathfrak{S} \text{ and } \operatorname{Im} S\tau\leq\eta\}$. But the author showed earlier [same Ann. **132** (1956), 121–129; MR **18**, 571] that if $E(\tau,s)=\sum(\operatorname{Im}S\tau)^s$ over $S\in\mathfrak{S}$, then $E(\tau,s)-2/(s-1)F$ is holomorphic in $\operatorname{Re}s\geq 1$. Thus by a theorem of Ikehara [see Wiener, *The Fourier integral and certain of its applications*, Cambridge Univ. Press, Cambridge, 1933; p. 127], $\lim_{\eta\to 0}\eta v_\eta(\tau)=2/F$. This result, with suitable majorant, permits the limit to be taken under the integral sign.

As a consequence, $\int_{\mathfrak{F}} g\,d\omega=F\lim_{y\to 0}\int_0^1 g(x+iy)\,dx$ in case the limit exists. *H. Cohn* (Tucson, Ariz.)

4831:

Heins, Maurice. On a notion of convexity connected with a method of Carleman. J. Analyse Math. **7** (1959), 53–77.

The theorem of Wiman-Heins asserts that a non-constant entire function f with bounded minimum modulus satisfies the growth condition

$$\lim_{r\to\infty} r^{-1/2}\log M(r;f)>0,\quad M(r;f)=\max_{|z|=r}|f(z)|.$$

The author first presents the following conjecture applying to a more general situation. For an $m>0$, let $\vartheta(r)$ denote the angular measure of the set $\{|f(z)|\leq m,\ |z|=r\}$. If $\vartheta(r)\geq\vartheta_0>0$ for $r>r_0>0$, then

$$\liminf_{r\to\infty} r^{-\beta}\log M(r;f)>0, \tag{1}$$

where $\beta=\pi/(2\pi-\vartheta_0)$. This conjecture is shown to be valid, and furthermore f is proved to be of very regular growth of order β, if the left hand side of (1) is finite.

It is remarked that the problem actually belongs to the theory of subharmonic functions, and it is so treated in the paper. In order to deal with this kind of problem in the analytic case, Carleman introduced a method in which the growth of an analytic function subject to certain boundary conditions is measured in terms of a quadratic integral norm, and information is obtained from a differential inequality satisfied by this norm. When one turns to the subharmonic situation difficulties are encountered due to the lack of smoothness conditions.

In the paper it is shown that the differential inequality of Carleman may be replaced by a convexity condition, and the principal result concerning subharmonic functions is stated in such terms. In addition to the aforementioned problem, the main theorem is shown to be applicable to other function-theoretic questions, e.g., to problems of Denjoy-Ahlfors type. *O. Lehto* (Helsinki)

4832:

de Branges, Louis. Some mean squares of entire functions. Proc. Amer. Math. Soc. **10** (1959), 833 839.

For sufficiently well behaved entire functions F of exponential type, it is true that, for most α,

$$\pi^{-1}\int_{-\infty}^{\infty}|F(t)|^2dt=\sum_{k=-\infty}^{\infty}|F(k\pi+\alpha)|^2.$$

The author generalizes this by introducing "pseudo" trigonometric functions, and an appropriately modified cardinal series expansion. Let $E(t)$ be entire, of exponential type, with no zeros in $y\geq 0$, and obeying

$$\int(1+t^2)^{-1}\log^+|E(t)|dt<\infty,$$

and $h(-\frac{1}{2}\pi,E)\leq h(\frac{1}{2}\pi,E)$ where

$$h(\theta,E)=\limsup_{r\to\infty} r^{-1}\log|E(re^{i\theta})|.$$

For any α, let $\Lambda(\alpha)$ be the (discrete) set of t for which $e^{i\alpha}E(t)$ is real. Let F be any entire function, also of exponential type and such that $\int(1+t^2)^{-1}\log|F(t)|dt<\infty$, and $F(iy)=O(E(i|y|))$ as $|y|\to\infty$. Then, for almost any real α,

$$\pi^{-1}\int\left|\frac{F(t)}{E(t)}\right|^2dt=2i\sum_{t\in\Lambda(\alpha)}\frac{|F(t)|^2}{G(t)},$$

where $G(t)=E(t)\overline{E'(t)}-\overline{E(t)}E'(t)>0$.

R. C. Buck (Madison, Wis.)

4833:

de Branges, Louis. Some Hilbert spaces of entire functions. Proc. Amer. Math. Soc. **10** (1959), 840–846.

Let H be a complex linear space of entire functions on which there is an inner product $\langle f,g\rangle$ and a norm $\|\ \|$ under which H becomes a hilbert space. Suppose that the evaluation functionals $f\to f(c)$ are continuous; that if $f\in H$, then $f^*\in H$ and $\|f\|=\|f^*\|$, where $f^*(z)=\overline{f(\bar z)}$; that if $f\in H$, then $g\in H$ and $\|f\|=\|g\|$, where $g(z)=(z-\bar c)f(z)/(z-c)$ and c is any non-real zero of f. The author shows that the norm must be given by

$$\|f\|=\left\{\int|f(t)|^2|E(t)|^{-2}\,dt\right\}^{1/2},$$

where E is an entire function, and that the functions in H must obey the uniform estimate

$$|f(z)|^2\leq(4\pi y)^{-1}\|f\|^2\{|E(z)|^2-|E(\bar z)|^2\}.$$

R. C. Buck (Madison, Wis.)

4834:

Shieh, Hui-chun. Sur la limitation d'une fonction holomorphe admettant deux valeurs exceptionnelles *B*. Sci. Record (N.S.) **3** (1959), 225–228.

The author continues the series of papers started by K. L. Hiong giving minor modifications of results of Valiron and Milloux concerning the distribution of values of meromorphic functions and their derivatives. A typical result is the following: If $f(z)$ is regular in $|z|<1$, and for some integer $k\geq 0$ and a positive λ we have

$$N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{(k)}-1}\right)\leq \lambda\log\frac{1}{1-r},\quad 0<v<1,$$

then

$$\log M(r,f)\leq \frac{H_k}{1-r}\left\{\log^+|f(0)|+\log\frac{1}{1-r}\right\},$$

where H_k depends on λ and k only. Proofs are only sketched, but since much stronger results are now known this does not matter. *W. K. Hayman* (London)

4835:

Jenkins, James A. On weighted distortion in conformal mapping. Illinois J. Math. **4** (1960), 28–37.

The author reproves the result of Oikawa [Kōdai Math. Sem. Rep. **9** (1957), 140–144; MR **19**, 1045] in which he gives precise upper and lower bounds for $|f'(z)|/(1+|f(z)|^2)$ as f varies through the class S of normalized univalent functions in the unit circle. He does this by first applying his General Coefficient Theorem [Jenkins, *Univalent functions and conformal mapping*, Springer, Berlin, 1958; MR **20** #3288] to find the exact region of values of the pair $(|f(z)|, |f'(z)|)$ for $f\in S$. This reduces the determination of bounds for any quantity of the form $F(|f(z)|, |f'(z)|)$ to a problem in calculus. *G. Springer* (Lawrence, Kans.)

4836:

Charzyński, Zygmunt. Fonctions univalentes inverses. Polynômes univalentes. Soc. Sci. Lettres Łódź. Bull. Cl. III **9** (1958), no. 7, 21 pp.

Let $\mathfrak{P}_M$ denote the class of polynomials of the form (1) $P(w)=w+C_2w^2+\cdots+C_Lw^L$, $L\leq M$, whose inverses are univalent in $|z|<1$, with $w=0$ corresponding to $z=0$. Let $G(X_2,\cdots,X_N,Y_2,\cdots,Y_N)$, $N\geq 2$, be a real-valued function having continuous first partial derivatives in a sufficiently large region, not all of which vanish simultaneously. The author derives the differential relations satisfied by an extremal polynomial in the family $\mathfrak{P}_M$ which maximizes the functional $G_P=G(\xi_2,\cdots,\xi_N,\eta_2,\cdots,\eta_N)$ with P as in (1) and $C_n=\xi_n+i\eta_n$, $N\leq M$. He also specializes these relations to the case where G is the real part of an analytic function of the complex variables $X_2+iY_2,\cdots,X_N+iY_N$. *G. Springer* (Lawrence, Kans.)

4837:

Merkes, E. P. On typically-real functions in a cut plane. Proc. Amer. Math. Soc. **10** (1959), 863–868.

The function f is said to be typically real in the domain D provided it is regular in D and, if z is in D, $\operatorname{Im} f(z)=0$ only in case $\operatorname{Im} z=0$. Rogosinski [Math. Z. **35** (1932), 93–121] showed that the function f, regular in $|z|<1$ and with $f(0)=0$ and $f'(0)=1$, is typically real in this domain only in case the function ϕ defined by $f(z)=(z/(1-z^2))\phi(z)$, $\phi(0)=1$, is real for real z and has positive real part, in $|z|<1$. On combining this result with theorems on the representation of functions as Stieltjes transforms and as continued fractions [Wall, *Analytic theory of continued fractions*, Van Nostrand, New York, 1948; MR **10**, 323; Chapter XV], the author obtains several theorems, including the following: The function f from the part of the complex plane exterior to the cut along the negative half of the real axis from -1 to $-\infty$, with $f(0)=0$ and $f'(0)=1$, is typically real in this domain only in case there exists a chain sequence a such that

$$f(z)=\frac{z}{1}+\frac{a_1z}{1}+\frac{a_2z}{1}+\cdots$$

$$(a_p=(1-g_{p-1})g_p;\ 0\leq g_{p-1}\leq 1).$$

He shows that this function f is univalent for $\operatorname{Re} z>-1$ and is convex in the direction of the imaginary axis for $|z|<1$. *H. S. Wall* (Austin, Tex.)

4838a:

Baker, I. N. Some entire functions with fixpoints of every order. J. Austral. Math. Soc. **1** (1959/61), 203–209.

4838b:

Baker, Irvine Noel. The existence of fixpoints of entire functions. Math. Z. **73** (1960), 280–284.

Let f be an entire function. Denote by f_p (p a natural number) the pth iterate of f. That is, $f_1=f$, and $f_{p+1}=f(f_p)$. A 'fixpoint' of f_p is a zero of $f_p(z)-z=0$; such a point is also referred to as a 'fixpoint of order p' of f. A point is called a 'fixpoint of exact order p' if it is a fixpoint of order p but not a fixpoint of lower order. In these two articles the author continues his earlier investigations [Math. Z. **71** (1959), 146–153; MR **21** #5743] concerning the existence of entire functions f having fixpoints of exact order p, for prescribed values of p. In the first paper it is shown that the elements of certain families of functions C_p have fixpoints of exact order p. It follows that all functions in $\bigcap C_p$ have fixpoints of exact order p, for every natural number p. All functions with Picard exceptional values belong to this intersection. In the second paper it is proved that for all entire functions, except first degree polynomials, there exist fixpoints of exact order p, for all except at most one value of p.

W. J. Thron (Boulder, Colo.)

4839:

Dragilev, M. M. On the convergence of the Abel-Gončarov interpolation series. Uspehi Mat. Nauk **15** (1960), no. 3 (93), 151–155. (Russian)

The Whittaker constant w is the lower bound of values c such that there are an entire $f(z)\not\equiv 0$ of exponential type 1, and a sequence $\{z_n\}$, $|z_n|\leq c$, with $f^{(n)}(z_n)=0$. The Gončarov constant γ is defined similarly with $f(z)$ regular in $|z|<1$ and $|z_n|\leq c/(n+1)$. Define w' as the lower bound of numbers c such that every $f(z)$ of exponential type 1 can be expanded in an Abel-Gončarov interpolation series with nodes z_n, $|z_n|\leq c$; and γ' similarly for $f(z)$ regular in $|z|<1$ with $|z_n|\leq c/(n+1)$. The inequalities $w'\leq w$ and $\gamma'\leq\gamma$ are evident; Evgrafov proved $w'=w$ [*Interpolyacionnaya zadača Abelya-Gončarova*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954; MR **16**, 1104]. The author

now proves $\gamma' = w'$, which lends additional plausibility to the conjecture that $\gamma = w$.

R. P. Boas, Jr. (Evanston, Ill.)

4840:

Makarova, L. Ya. On domains convex with respect to a class of polynomials. Uspehi Mat. Nauk **14** (1959), no. 4 (88), 173–178. (Russian)

The author studies the class Σ_0 of domains in (z, w)-space defined by a finite number of conditions $P_k(z, w) \in D_k$ $(k = 1, \cdots, n)$, where each P_k is a polynomial and each D_k a domain bounded by a finite number of Jordan curves. She further considers the class Σ_1 of domains bounded by subsets of a finite number of hypersurfaces $P_k(z, w) = c_k(t)$ $(k = 1, \cdots, n)$, where each P_k is a polynomial and each $c_k(t)$ a Jordan curve. She proves by an example that Σ_0 is a proper subset of Σ_1. Finally, the author gives rather simple sufficient conditions for domains of the class Σ_1 to be convex with respect to the class of polynomials. *H. Tornehave* (Copenhagen)

SPECIAL FUNCTIONS

4841:

Butler, R. On the evaluation of $\int_0^\infty (\sin^m t)/t^m dt$ by the trapezoidal rule. Amer. Math. Monthly **67** (1960), 566–569.

Fettis [Math. Tables Aids Comput. **9** (1955), 85–92; MR **17**, 302] has pointed out that the error in computing $\int_0^\infty f(t)dt$ by the trapezoidal rule can be expressed as an infinite series involving the Fourier cosine transform of f. The author remarks that in case the Fourier transform vanishes outside a finite interval, the trapezoidal rule will give exact results if the subdivision is sufficiently fine. Specifically, he proves that

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^m dt = \left[\frac{1}{2} + \sum_{r=1}^{m} \left(\frac{\sin hr}{hr}\right)^m\right]$$

if $m = 1, 2, \cdots$; $h < 2\pi$ if $m = 1$, and $h \leq 2\pi/m$ if $m = 2, 3, \cdots$. He also provides alternative expressions for the sum on the right-hand side. *A. Erdélyi* (Pasadena, Calif.)

4842:

Popov, Blagoj Sazdov. A note concerning the integrals involving the derivatives of Legendre polynomials. Rend. Sem. Mat. Univ. Padova **29** (1959), 316–317.

Short proof of a formula established by S. K. Chatterjee [same Rend. **27** (1957), 144–148; MR **19**, 850].

P. Henrici (Los Angeles, Calif.)

4843:

Fieldhouse, Martin. A function with an infinity of saddle points. Eureka, no. 21 (1958), 10–11.

L'auteur étudie les solutions en x de l'équation $M(a, b, x) = 0$, $M(a, b, x)$ désignant la fonction hypergéométrique, a et b étant donnés; il n'existe, dit-il, qu'un nombre fini de valeurs positives de x solutions de l'équation.

Il donne un graphique. *R. Campbell* (Caen)

ORDINARY DIFFERENTIAL EQUATIONS

See also 4912, B5123, B5339.

4844:

Hoheisel, Guido. ★Gewöhnliche Differentialgleichungen. 6te, neubearbeitete und erweiterte Aufl. Sammlung Göschen, Bd. 920. Walter de Gruyter & Co., Berlin, 1960. 128 pp. DM 3.60.

The fifth edition [1956] was listed in MR **18**, 127. Contents: I. First order (vector) equations (general; singular solutions and families of curves; behavior near a singular point; theorems in the large; dependence on initial values and parameters). II. Linear equations (general; Green's function; adjoint equation; constant coefficients). III. Boundary value problems (general; eigenvalues; Sturm-Liouville systems; asymptotic computation of eigenvalues and eigenfunctions; non-Liouville systems; periodicity; completeness of the system of eigenfunctions). IV. Oscillation (Sturm theorems; distribution of zeros; criteria for oscillation and non-oscillation).

4845:

Hájek, Otomar. Singularities of differential equations. I, II. Pokroky Mat. Fys. Astr. **1** (1956), 551–559; **2** (1957), 137–144. (Czech)

By a 'singular point' of the equation (1) $dy/dx = F(x, y)$ (x, y being real, F a continuous function), the author means a point at which uniqueness does not hold. From the results proved follows for instance this assertion: let S be the set of singular points, S^+ [S^-] the set of points at which uniqueness fails with increase [decrease] in x, U^* the derived set of a set U; if $S \cap S^{+*} \cap S^{-*}$ is a denumerable set, then S has measure 0.

A characterization is given of sets of functions $\{f_\alpha\}$ with the property that there exists an equation (1) of which all the f_α are solutions. An example is constructed in which the closure of the set S is the y-axis. A large number of other results are proved, many of which are known or evident. *J. Kurzweil* (RŽMat **1959** #337)

4846:

Filippov, A. F. Differential equations with discontinuous right-hand side. Mat. Sb. (N.S.) **51 (93)** (1960), 99–128. (Russian)

Systems of ordinary differential equations with discontinuous right-hand side have been studied from the point of view of generalizing the existence theorems and also from the point of view of obtaining qualitative information about the solutions. In seeking to unify both points of view, the author introduces two definitions (which are shown to be equivalent) of the solutions of a system of ordinary differential equations with discontinuous right-hand side; and studies existence, prolongation of solutions, uniqueness, continuous dependence of solutions on parameters of such systems. Let the system be written, in vector form, as follows: $dx/dt = f(t, x)$, where $x = (x_1, \cdots, x_n)$, $f = (f_1, \cdots, f_n)$; and for each i the real-valued measurable function $f_i(t, x_1, \cdots, x_n)$ is defined almost everywhere on a set Q (open or closed), such that there is, for each closed and bounded subset D of Q, an almost everywhere finite function $A(t)$ for which

$|f_i(t, x_1, \cdots, x_n)| \leq A(t)$. According to the first definition, a vector-valued function $x(t)$ is said to be a solution of such a system on an interval (t_1, t_2) provided that $x(t)$ is absolutely continuous on this interval and that, for almost every t on this interval, it is true that for every $\delta > 0$ the vector $dx(t)/dt$ belongs to the smallest convex closed set (in n-dimensional space) which contains all the values of the vector-valued function $f(t, x')$, when x' varies over almost all the δ-neighborhood of the point $x(t)$ in the x-space (in symbols,

$$\frac{dx(t)}{dt} \in \prod_{\delta>0} \prod_{\mu N=0} \operatorname{konv} f(t, U(x(t), \delta) - N),$$

where μ is Lebesgue measure, $U(x, \delta)$ is the δ-neighborhood of x, and konv E is the smallest closed convex set containing the set E). *J. B. Diaz* (College Park, Md.)

4847:

Hull, T. E.; Luxemburg, W. A. J. Numerical methods and existence theorems for ordinary differential equations. Numer. Math. **2** (1960), 30–41.

This paper is concerned with existence theorems for single differential equations $x' = f(t, x)$, $x(t_0) = x_0$. The original proof of the Cauchy-Lipschitz theorem is based on the convergence of Euler's polygon method $y_{n+1} = y_n + hf(t_n, y_n)$, $y_0 = x_0$. Here, the authors investigate the feasibility of using more sophisticated methods, in particular multi-step methods

$$\sum_{\nu=0}^{k} \alpha_\nu y_{n+\nu} = h \sum_{\nu=0}^{k} \beta_\nu f(t_{n+\nu}, y_{n+\nu}),$$

for the same purpose. They show that the Cauchy-Lipschitz theorem, and a more general theorem by Osgood [Monatsh. Math. **9** (1898), 331–345], can be established by proving convergence of any multi-step method which is stable and consistent in a sense defined by the authors. Also a theorem by Peano [Math. Ann. **37** (1890), 182–228] is established with the aid of suitably restricted multi-step methods.

The authors present by way of introduction a simple and lucid proof of a theorem by Dahlquist [Math. Scand. **4** (1956), 33–53; MR **18**, 338] which states necessary and sufficient conditions for stability, and also gives a characterization of consistent multi-step methods.

Walter Gautschi (Oak Ridge, Tenn.)

4848:

Karaseva, T. M. On an "exact estimate" of the multipliers of second order differential equations with periodic coefficients. Dokl. Akad. Nauk SSSR **121** (1958), 34–36. (Russian)

Take the Hill equation (1) $y'' + p(x)y = 0$, where p has period T. There are two solutions y_1, y_2 which upon addition of T are just multiplied say by ρ_1, ρ_2. Here the ρ_i satisfy an equation $\rho^2 - 2A\rho + 1 = 0$, and we have stability when and only when $|A| < 1$.

Now Lyapunov studied the solutions of (2) $y'' + \lambda p(x)y = 0$ [C. R. Acad. Sci. Paris **128** (1899), 910–913, 1085–1089] and showed that the λ axis contains an infinity of intervals in which $|A(\lambda)| < 1$. These are the stability zones and they are numbered 1, 2, 3, $\cdots$, in the direction of increasing λ. Between 1 and 2, 2 and 3, $\cdots$, there is an instability zone, and these are also numbered 1, 2, $\cdots$. If $\lambda = 1$ is in the stability or instability zone k, then $p(x)$ is said to be in that zone. This paper improves results of Yakubovič [Dokl. Akad. Nauk SSSR **87** (1952), 345–348; Prikl. Mat. Meh. **18** (1954), 533–546; MR **14**, 646; **16**, 821] and Burdina [Dokl. Akad. Nauk SSSR **93** (1953), 603–606; MR **15**, 624].

Let Q_a denote the set of all real, square integrable functions on $[0, T]$ extended periodically, such that $T\int_0^T q^2(x)dx = \alpha$, $\int_0^T q(x)dx = 0$. Let $P_a = \{p(x) | p = q', q \in Q_a\}$.

Theorem 1: If $p \in P_a$ is in the nth instability zone then $n < [2\alpha^{1/2}/\pi]$, and for every n in that range there is such a $p \in P_a$. Under these circumstances

$$\rho_{1,2} \leq \exp\left\{\alpha \sqrt{\left(\frac{2n\alpha^{1/2}}{\pi} - n^2\right)}\right\}.$$

These are also best possible results. Theorem 2: If $p \in P_a$ and $\alpha < \frac{1}{4}n^2\pi^2$ then all solutions of (1) are bounded. [Additional reference: Kreĭn, Prikl. Mat. Meh. **19** (1955), 641–680; MR **17**, 1088.] *S. Lefschetz* (Princeton, N.J.)

4849:

Roĭtenberg, Ya. N. On the accumulation of perturbations in non-stationary linear systems. Dokl. Akad. Nauk SSSR **121** (1958), 221–224. (Russian)

A certain system is reducible by a change of coordinates to the n-vector form

$$\dot{z} + a(t)z = X(t).$$

The standard solution of this equation is used to obtain upper bounds for the absolute values of the components of z. [References: B. V. Bulgakov, *Prikladnaya teoriya giroskopov*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1939; pp. 79, 149; Dokl. Akad. Nauk SSSR **51** (1946), 343–345; MR **8**, 70; Bulgakov and Kuzovkov, Prikl. Mat. Meh. **14** (1950), 7–12; MR **11**, 597; Bliss, *Mathematics for exterior ballistics*, Wiley, New York, 1944; MR **6**, 24; H. S. Tsien, *Engineering cybernetics*, McGraw-Hill, New York, 1954.] *S. Lefschetz* (Princeton, N.J.)

4850:

Olech, C. Estimates of the exponential growth of solutions of a second order ordinary differential equation. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 487–494. (Russian summary, unbound insert)

Let $a(t)$, $b(t)$ be piecewise continuous for $t \geq 0$ and satisfy $a_1 \leq a(t) \leq a_2$, $b_1 \leq b(t) \leq b_2$, where a_i, b_i are real constants such that $c^2 - d < 0$ if $a_1 \leq c \leq a_2$, $b_1 \leq d \leq b_2$. It is shown how to find the "best" constants ρ_1, ρ_2 depending only on a_1, a_2, b_1, b_2 with the property that, if $x = x(t)$ is any solution of (1) $x'' + 2a(t)x' + b(t)x = 0$, then

$$\limsup |x(t)| \exp[-(\rho_1 - \varepsilon)t] > 0$$

and

$$\limsup |x(t)| \exp[-(\rho_2 + \varepsilon)t] < \infty,$$

as $t \to \infty$, for every $\varepsilon > 0$. (The case where $c^2 - d \geq 0$ if $a_1 \leq c \leq a_2$, $b_1 \leq d \leq b_2$ follows from Olech, same Bull. **7** (1959), 319–326 [MR **21** #7326].) In the proof, the author considers the linear binary system for (2) $(u(t), v(t)) = (x(t)e^{-\rho t}, x'(t)e^{-\rho t})$ and simple auxiliary binary linear systems with discontinuous, piecewise constant coefficients. In terms of solutions of these auxiliary systems, he constructs functions $W_1(u, v)$, $W_2(u, v)$, depending on ρ, such that for any solution $x = x(t)$ of (1), the functions

(2) have the properties that $W_1(u(t), v(t))$ is non-decreasing for some ρ and $W_2(u(t), v(t))$ is non-increasing for some ρ.

P. Hartman (Baltimore, Md.)

4851:

Martirosyan, R. M. A general Sturm-Liouville problem. Akad. Nauk Armyan. SSR. Dokl. **29** (1959), 49–58. (Russian. Armenian summary)

The author considers the effect on the spectrum of $T_0 = -d^2/dx^2$ (on L_2 $(-\infty, \infty)$) of the addition of T_1, where $T_1 u = p(x)Lu + \int_{-\infty}^{\infty} K(x,t)u(t)dt$. Here L is a bounded operator on L_2, $p \in L_2$ is bounded, and $K(x,t)$ is a kernel of Hilbert-Schmidt type. In theorem 1 the author shows that the continuous spectrum of $T = T_0 + T_1$ is the positive semi-axis, and that the point spectrum can only have accumulation points in the positive semi-axis. In succeeding theorems the author obtains some more explicit results under more detailed assumptions. [Cf. #4873 below.]

R. R. D. Kemp (Kingston, Ont.)

4852:

Fujita, Hiroichi. Oscillation represented by the third order differential equation. II. Proc. Fac. Engrg. Keio Univ. **11** (1958), 94–101.

From the third order differential equation

$$x''' + F(x'', x', x) = 0, \quad x' = dx/dt,$$

one secures the system $x' = y$, $y' = z$, $z' = -F(x, y, z)$ from which it follows that $dx/dy = y/z$ and $dz/dy = -F(x, y, z)/z$. The author, in order to remove z from the denominators, sets $E_1 = \frac{1}{2}(z^2 + y^2)$, $E_2 = \frac{1}{2}(y^2 + z^2)$ to secure the system $dE_1/dx = x + z$, $dE_2/dy = y - F(x, y, z)$. Advantages of discussing $x''' + F(x'', x', x) = 0$ in the $x - E_1$ and $y - E_2$ planes are given, and this approach is used in a study of a Colpittz oscillator equation $x''' + Ax'' + x' + (x - \varepsilon x^3) = 0$. This paper is independent of the author's earlier paper of the same title in same Proc. **8** (1955), 61–67 [MR **19**, 884].

W. R. Utz (Columbia, Mo.)

4853:

Movšovič, S. M. Determination of the domain of attraction of some singular points of higher order. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1959**, no. 6, 3–11. (Russian)

If x is an n-vector with components x_i, let x^k denote the vector with components x_i^k, $k > 1$. The paper deals with systems of the very special type

$$\dot{x} = Ax^k + f(x; t) \tag{1}$$

with A a constant matrix, and the associated system of the first approximation

$$\dot{x} = Ax^k. \tag{2}$$

Let $T(R)$ be a sphere of radius R centered at the origin. It is called the spherical region of influence of the critical point provided it contains no complete path but every $T(R+\varepsilon)$ does contain such a path (origin always excepted). The determination of $T(R)$ rests upon this result of Krassovskiĭ [Prikl. Mat. Meh. **18** (1954), 513–532; MR **19**, 473]. A necessary and sufficient condition in order that any T contain no complete path of $\dot{x} = X(x)$, $X(0) = 0$, is that there exist a Liapunov function in T, of class C^1, whose derivative $\dot{V}$ along the paths is < 0. The first result is the following. Theorem 1: If k is even and the matrix A is of rank n or $n-1$, then no finite sphere T contains a complete path of (2). If p is odd, same conclusion under the condition that there exist numbers c_i such that $(c_i a_{ij} + c_j a_{ji})$ is the matrix of a positive definite quadratic form.

Consider now the system (1) where in some $T(R)$, $\|f\| < c_R \sum x_j^k$ with c_R independent of $(x; t)$. Let μ be the number of components of f that are $\not\equiv 0$. Let $c_{\max}$ be the largest $|c_i|$ satisfying $\sum c_i a_{ij} = 1$, where A is of rank n. Theorem 2: If in $T(R)$ we have $c_R < 1/\mu c_{\max}$, then $T(R)$ contains no complete trajectory. If this holds for all R then the whole space is the sphere of influence of the origin. A somewhat more complicated analogous result holds for k odd. [Additional reference: Papuš, Mat. Sb. (N.S.) **38** (**80**) (1956), 337–358; MR **17**, 1085.]

S. Lefschetz (Princeton, N.J.)

4854:

Kim, Če Džen. Determination of the region of influence of an asymptotically stable position of equilibrium. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1959**, no. 2, 3–14. (Russian)

Take an n-vector system

$$\dot{x} = Ax + X(x), \quad X(0) = 0, \tag{1}$$

where A is a constant matrix and X is of class C^1 with bounded second partials in some region $G: |x_s| \leq H$ $(s = 1, 2, \cdots, n)$. If G_0 is a region containing the origin 0, the sphere $T(R): \sum x_i^2 \leq R^2$ is said to be in the region of attraction of the (assumed) asymptotically stable origin relative to G_0 if all the paths initiating in $T(R)$ remain in G_0 and $\to 0$ as $t \to +\infty$. Estimates for the region of attraction have been given by Nemyckiĭ [Dokl. Akad. Nauk SSSR **101** (1955), 803–804; Trudy Moskov. Mat. Obšč. **5** (1956), 455–482; MR **17**, 263; **22** #802], Malkin [Prikl. Mat. Meh. **16** (1952), 495–499; MR **14**, 170], Atrašenok [Vestnik Leningrad Univ. **9** (1954), no. 8, 79–106; MR **17**, 1088], and Alzerman [Avtomat. i Telemeh. **8** (1947), 20–29; MR **12**, 27]. A new estimate is provided in the present paper, resting above all on the construction of a Liapunov function which is a definite positive quadratic form with constant coefficients. [Additional reference: Gorbunov, Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **1950**, no. 10, 19–26; MR **14**, 751.]

S. Lefschetz (Princeton, N.J.)

4855:

Pliss, V. A. The necessary and sufficient conditions for the stability in the large of a homogeneous system of three differential equations. Dokl. Akad. Nauk SSSR **120** (1958), 708–710. (Russian)

Let $\dot{x} = y - ax - f(x)$, $\dot{y} = z - bf(x)$, $\dot{z} = -cf(x)$, where a, b, c are constants, $ab > c$, $b > 0$, $c > 0$, and $f(x)$ is continuous and smooth enough so that every solution is unique, and $xf(x) > 0$ for $x \neq 0$, $f(0) = 0$. The author proves that the trivial solution is stable in the large if and only if

$$\limsup_{x \to +\infty} \left[f(x) + \int_0^x f(t)dt \right] = +\infty,$$

$$\liminf_{x \to +\infty} \left[-f(x) + \int_0^x f(t)dt \right] = +\infty.$$

The proof of the sufficiency depends on previous results

of the author [same Dokl. **103** (1955), 17–18; MR **17**, 152] and of Tuzov [Vestnik Leningrad. Univ. (1955), no. 2, 43–70; MR **16**, 1025].

H. A. Antosiewicz (Los Angeles, Calif.)

4856:

Yakubovič, V. A. On boundedness and stability in the large of the solutions of some non-linear differential equations. Dokl. Akad. Nauk SSSR **121** (1958), 984–986. (Russian)

This is a continuation of the work in same Dokl. **117** (1957), 44–46 [MR **19**, 549]. The notations are the same. The system is

$$\dot{x} = Ax+\varphi(\sigma)a, \quad \dot{\sigma} = b\cdot x-\rho\varphi(\sigma), \tag{1}$$

where A is an n-matrix whose characteristic roots λ_j have negative real parts; x, a, b are n-vectors and the rest are scalars with $\rho>0$; in particular σ and $\varphi(\sigma)$ are the control variable and function. One condition for stability in the large was

$$\rho+b\cdot A^{-1}a > 0. \tag{2}$$

Theorem: Let $|\varphi(\sigma)|\leq\varphi_0$ for all σ, and let (2) hold. Then every solution of (1) exists on $[0,+\infty]$ and is bounded as $t\to+\infty$. Moreover if $\|e^{At}\|<\alpha e^{-\beta t}$, α and $\beta>0$, then for $t\geq 0$:

$$\|x(t)\| \leq \alpha e^{-\beta t}\|x(0)\|+\frac{\alpha\rho_0}{\beta}(1-e^{-\beta t})\|a\|,$$

$$\|\sigma(t)\| \leq \sigma(0)+2\max_{t\geq 0}|b\cdot A^{-1}x|.$$

Further results are given when one of the λ_j is zero. Under certain conditions stability in the large is possible whatever $\varphi(\sigma)$. Not so however when two or more of the λ_j are zero. [References: Lur'e, *Some nonlinear problems in the theory of automatic regulation*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951; MR **15**, 707; V. A. Yakubovič, Vestnik Leningrad. Univ. **12** (1957), no. 19, 172–176; Aĭzerman, *Lectures on the theory of automatic regulation*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1956; MR **18**, 981; p. 220; Pliss, #4855 above.]

S. Lefschetz (Princeton, N.J.)

4857:

Lee, Shen-ling. Topological structure of integral curves of the differential equation

$$y' = \frac{a_0x^n+a_1x^{n-1}y+\cdots+a_ny^n}{b_0x^n+b_1x^{n-1}y+\cdots+b_ny^n}.$$

Acta Math. Sinica **10** (1960), 1–21. (Chinese. English summary)

The author first deals with the case where the differential equation has m ($\leq n+1$) solutions which are straight lines through the only critical point, the origin in xy-plane. The topological structure of the totality of integral curves in the neighborhood of the critical point is characterized and classified by those in the neighborhood of each of such straight lines. Elaborate formulas are worked out for the number of the classes when n is an arbitrary positive integer.

S. S. Shu (Lafayette, Ind.)

4858:

Zverkin, A. M. Dependence of the stability of solutions of linear differential equations with lagging argument upon the choice of the initial moment. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1959**, no. 5, 15–20. (Russian)

Consider the equation (1) $y'(x)=\int_0^{\Delta(x)} y(x-s)dr(x,s)$, $\Delta(x)\geq 0$, with the initial conditions specified as follows. If A is a given number let $T(A)=\inf_{A\leq x<\infty}(x-\Delta(x))$ and let $E_A=[T(A),A]$. The initial values of $y(x)$ are then given by a function $\varphi(x)$ defined on E_A and $y(x-s)=\varphi(x-s)$, $x-s\leq A$. The author assumes that $r(x,s)$, $\varphi(x)$ satisfy conditions which assure the existence and uniqueness of a solution of (1). [See A. D. Myškis, *Linejnye differencial'nye uravneniya s zapazdyvajuščim argumentom*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1951; MR **14**, 52.] If the solution $y=0$ of (1) is stable with respect to perturbations of the initial values on E_A, then it is stable with respect to perturbations on E_{A_1}, $A_1\leq A$. However, the author gives two examples to show that this is not the case for $A_1>A$. On the other hand, he shows that if $r(x,s)$ is nondecreasing in s for every fixed x, $A\leq x<\infty$, then the property of stability does not depend upon the number A. Some other results of this type are given for the equation $y'(x)=a(x)y(x)+b(x)y(x-\tau(x))$, $\tau(x)\geq 0$.

J. K. Hale (Baltimore, Md.)

4859:

Él'sgol'c, L. Ė. On the theory of stability of differential equations with retarded argument. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1959**, no. 5, 65–71. (Russian)

This paper discusses various problems associated with differential-difference equations of the form (1) $\dot{x}(t)=F(t,x(t),x(t-\tau(t)))$ or equations of neutral type (2) $\dot{x}(t)=F(t,x(t),x(t-\tau(t)),\dot{x}(t-\tau(t)))$, where x, F are n-vectors, τ is a scalar, $\tau(t)\geq 0$ is continuous. There also may be more than one lag in the equations. The set E_{t_0} is defined as in the preceding review, and the initial values of x are given on E_{t_0} by a function φ. The reviewer assumes (even though it is not discussed in the paper) that the author has a well-defined concept of a solution $x_\varphi(t)$ of (1) and (2), and that F and τ are such that an existence and uniqueness theorem holds. Krasovskiĭ [*Nekotorye zadači teorii ustoĭčivosti dviženiya*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1959; MR **21** #5047] has discussed the stability question for (1) using a Lyapunov functional and a metric which depends only on $x_\varphi(t)$. If one uses a proper metric involving both $x_\varphi(t)$ and $\dot{x}_\varphi(t)$, the author remarks that the results of Krasovskiĭ remain valid for (2), and gives an example to illustrate this fact. He also proves a theorem concerning the "equivalence" of the stability problem for an nth order equation and for a system of n first order equations.

J. K. Hale (Baltimore, Md.)

4860:

Wang, Lian. On the equivalence problem of differential equations and difference-differential equations in the theory of stability of the first critical case. Acta Math. Sinica **10** (1960), 104–124. (Chinese. English summary)

The author extends the stability (or instability) criterion for the critical point at the origin of the system

of differential equations in the vector form $dy/dt = f(y)+g(y)$, where $y=y(t)$, to the system of difference-differential equations of the form $dy/dt = f(y(t))+g(y(t-\delta(t)))$, provided that $0 \leq \delta(t) \leq \Delta$ for some positive Δ. He is concerned with the case where both f and g are regular at the origin and the matrices of the linear parts of the system (due to f and g respectively) are such that the first row are all zeroes. *S. S. Shu* (Lafayette, Ind.)

4861:

Chin, Yuan-shun. Unconditional stability of systems with time-lags. Acta Math. Sinica **10** (1960), 125–142. (Chinese. English summary)

The necessary and sufficient conditions for all solutions of the linear system in the vector form $dx/dt = Ax(t)+Bx(t-\tau)$, where A and B are constant $n \times n$ matrices, to be asymptotically stable for all $\tau \geq 0$, are that (1) all the roots λ of the equation $|A+B-\lambda I|=0$ have negative real parts, where I is the identity matrix; and (2) $|A+Be^{iw}-iyI| \neq 0$ for all real values of w and y $(\neq 0)$. Hence the question of solving a transcendental equation is reduced to dealing with algebraic ones. The cases $n=1$ and $n=2$ are discussed in detail and the conditions on A and B for stability are worked out explicitly for these cases. *S. S. Shu* (Lafayette, Ind.)

4862:

Él'sgol'c, L. Ė. Some properties of periodic solutions of linear and quasi-linear differential equations with retarded argument. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1959**, no. 5, 229–234. (Russian)

The author first considers equations of the form

$$(1) \qquad x^{(n)}(t)+\sum_{0}^{n-1}[a_k x^{(k)}(t)+b_k x^{(k)}(t-\tau)] = f(t),$$

or

$$(2) \qquad \sum_{0}^{n}[a_k x^{(k)}(t)+b_k x^{(k)}(t-\tau)] = f(t),$$

where $\tau > 0$, $a_n, b_n \neq 0$. The first two theorems consider "proper frequencies" of (1) or (2); that is, purely imaginary roots of the associated characteristic equation for $f=0$. It is shown that (1) can have no more than n proper frequencies, while (2) either has no more than n, or, in an exceptional case when a certain polynomial relation becomes an identity, a countable infinity of proper frequencies. In a corollary, it is pointed out that if f is periodic, resonance can occur in k frequencies $(k \leq n)$, or even in an infinite number of frequencies, in the exceptional case of (2).

Next, assuming f is periodic in t of period T, an existence theorem for a periodic solution of (1) is given as follows. (Here Φ denotes an initial function defined over a suitable initial interval.) If there exists a uniformly asymptotically stable solution $X_1(t)$ of (1), such that for $t \geq T_1$, $\|X_1(t)\| \leq N$, and the domain of attraction of the solution $X_1(t)$ contains all solutions whose initial functions satisfy $\|\Phi(t)\| \leq N+\varepsilon$, on the initial interval, $\varepsilon > 0$, then there exists a periodic solution, with initial function Φ_1, and $\|\Phi_1(t)\| \leq N+\varepsilon$ on the initial interval. The proof uses Schauder's fixed point theorem.

The next theorem then shows that any periodic solution may be regarded as simply the translation, along the t-axis, through some multiple of T, of a suitable initial function. This theorem, together with the existence theorem, then implies that (1) has a unique periodic solution (unique among those solutions whose initial functions satisfy $\|\Phi(t)\| \leq N+\varepsilon$), and the period of this unique solution is T.

Some generalizations of the results on periodic solutions are discussed, including vector equations with several lags, which are continuous periodic functions of T. The author points out that to apply Schauder's theorem to (2), or its generalization, requires some additional unspecified restrictions. *A. Stokes* (Baltimore, Md.)

4863:

Takahashi, Ken-ichi. Über eine erweiterte asymptotische Darstellung der Lösung eines Systems von linearen homogenen Differentialgleichungen, welche von zwei Parametern abhängen. Tôhoku Math. J. (2) **10** (1958), 172–193.

The author has succeeded in making further progress in the study of solutions of ordinary differential equations involving two large parameters λ and μ. More precisely, he has obtained asymptotic series solutions of a matrix equation $dY/dx = \lambda^m \mu^n A(x, \lambda, \mu)Y$ of second order, where asymptotically

$$(1) \qquad A(x, \lambda, \mu) \sim \sum_{r,s=0}^{\infty} A_{rs}(x)\lambda^{-r}\mu^{-s}$$

and $A_{00} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. This has been accomplished by generalizing the concept of asymptotic series expansions.

{Unfortunately the paper is marred by a number of misprints and a lack of precision in presenting the new and important generalization of asymptotic expansions. For instance, it is not explained that the asymptotic expansion indicated in the second formula of (2.5) is to be taken in the old sense, that of Poincaré, while the expansion (1) above and that in (ix), p. 178, has an extended meaning tardily indicated by formula (2.12). Furthermore, it would have been easier for the reader if it had been pointed out that the k in (iv), p. 174, can take on only a certain fixed and finite number of possible values; that the value of K in the first formula of (C), p. 175, depends in general on N; and that the bound on $f_r(x, \lambda, \mu)$ usually will depend upon r.}

H. L. Turrittin (Minneapolis, Minn.)

PARTIAL DIFFERENTIAL EQUATIONS

See also B5128.

4864:

Solomon, Liviu. Sur les conditions aux limites naturelles. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **9** (1957), 107–120. (Romanian. Russian and French summaries)

Consider the solution of the boundary value problem consisting of the equation $Au=f$, where A is a linear differential operator of order $2k$, subject to the homogeneous boundary conditions $B_i u = 0$ $(i=1, 2, \cdots, k)$, by means of the Rayleigh-Ritz method, where the solution u_0 is obtained in the form of a series $u_0 = \lim_{n\to\infty} \sum_1^n a_j \varphi_j$, and the Ritz "coordinate functions" satisfy all the homogeneous boundary conditions, i.e., $B_i \varphi_j = 0$ $(i=1, 2, \cdots, k;$

$j=1, 2, \cdots, n, \cdots$). The author extends a result of S. G. Mihlin [*Pryamye metody v matematičeskoĭ fizike*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950; MR **16**, 41] which gives a sufficient condition for reducing the number of boundary conditions to be required of the "coordinate functions" φ_j. Three applications of the author's criterion are given.

J. B. Diaz (College Park, Md.)

4865:

Rasulov, M. L. The residue method for solution of mixed problems for differential equations and a formula for expansion of an arbitrary vector-function in fundamental functions of a boundary problem with a parameter. Mat. Sb. (N.S.) **48 (90)** (1959), 277–310. (Russian)

The following initial value-boundary value problem is considered:

$$\frac{\partial^q u}{\partial t^q} = \sum_{\substack{k\le q-1\\ mk+l\le p}} A_{kl}(x)\frac{\partial^{k+l}u}{\partial t^k \partial x^l} + f(x,t) \quad (0 \le t \le T),$$

$$\sum_{\substack{k\le q\\ l\le p-1}} \partial_{kl}u = 0, \quad \left.\frac{\partial^k u}{\partial t^k}\right|_{t=0} = \Phi_k(x) \quad (k = 0, \cdots, q-1),$$

in which $\partial_{kl}u = P_{kl}\partial^{k+l}u/\partial t^k \partial x^l|_{x=a} + Q_{kl}\partial^{k+l}u/\partial t^k \partial x^l|_{x=b}$. Here m, p, q are natural numbers with $p=mq$, A_{kl} is an $n\times n$ matrix, f, Φ_k, u are n-vectors defined for $x\in[a,b]$ and $t\in[0,T]$, P_{kl} and Q_{kl} are $np\times n$ constant matrices, and, in the boundary condition, $mk+1\le p$ if $n=1$. Sufficient conditions are found under which a sufficiently smooth solution of this problem has the representation

$$-(2\pi i)^{-1}\sum_\nu \int_{c_\nu} \lambda^{m-1}e^{t\lambda^m}d\lambda\Big\{\int_a^b G(x,\xi,\lambda)[F_0(\xi,\Phi,\lambda^m) + \int_0^t e^{-\tau\lambda^m}f(\xi,\tau)d\tau]d\xi + \Delta_0(x,\Phi,\lambda)\Big\},$$

where

$$F_0 = \sum_{k=0}^{q-1}\lambda^{m(q-1-k)}\Phi_k - \sum_{\substack{1\le k\le q-1\\ mk+l\le p}} A_{kl}\frac{d^l}{dx^l}\Big[\sum_{j=0}^{k-1}\Phi_j\lambda^{m(k-1-j)}\Big],$$

c_ν is a closed contour enclosing just one pole of the integrand, Δ_0 is the solution of the following related spectral problem:

$$\sum_{k\le q-1}\lambda^{mk}A_{kl}d^lv/dx^l - \lambda^{mq}v = F_0,$$

$$\sum_{\substack{k\le q\\ l\le p-1}}\lambda^{mk}\partial_{kl}v = N_0(\Phi,\lambda^m),$$

and G is its Green's function. The theory is based upon a theorem for the expansion of an arbitrary vector function in fundamental solutions of a two point boundary value problem for a normal system of ordinary equations containing a parameter. The results are extended to systems with time dependent coefficients.

N. D. Kazarinoff (Moscow)

4866:

Boigelot, A. Comportement asymptotique du noyau spectral de $-\Delta$ sous des conditions aux limites de Dirichlet-Neumann dans un ouvert donné. Bull. Soc. Roy. Sci. Liège **28** (1959), 175–179.

Gårding's formula [Conferences on partial differential equations, University of Kansas, 1954; p. 8] for the asymptotic behavior of the spectral function of the Laplacian operator $-\Delta$ with Dirichlet-Neumann boundary conditions in an open region Ω is established using a simple Tauberian theorem. If E_λ is a spectral projection, then for f in $L_2(\Omega)$ one has $E_\lambda f = \int_\Omega E_\lambda(x_0,x)f(x)dx$, where $E_\lambda(x_0,x)\sim(4\pi)^{-n/2}(\lambda^{n/2}/\Gamma(\frac{1}{2}n+1))$. The Tauberian theorem employed is the following: If $\alpha(\lambda)$ is non-decreasing and such that $\lim_{t\to 0} t^\gamma \int_0^\infty e^{-\lambda t}d\alpha(\lambda)=C$, then $\alpha(\lambda)\sim C\lambda^\gamma/\Gamma(\gamma+1)$ as $\lambda\to\infty$.

G.-C. Rota (Cambridge, Mass.)

4867:

Martynyuk, A. E. Variational methods in boundary value problems for weakly elliptic equations. Dokl. Akad. Nauk. SSSR **129** (1959), 1222–1225. (Russian)

The author defines for the case of two independent variables (x, y) weak ellipticity of a linear partial differential operator Au of any order $m\ge 2$ and considers the equation $Au=f(x,y)$ in a region bounded by a piecewise smooth closed curve S. For $m=2l$ the boundary data are the vanishing of u and its partial derivatives of orders $1, \cdots, l-1$ on S; for $m=2l-1$, the vanishing on S of u, δu, and the partial derivatives of δu of orders $1, \cdots, l-2$, where $\delta u = \partial u/\partial x + \partial u/\partial y$. There is a unique solution and the Ritz-Krylov method in Hilbert space for finding it converges, for m even by a theorem of Mihlin [Uspehi Mat. Nauk **5** (1950), no. 6 (40), 3–51; MR **16**, 40], and for m odd of the author [same Dokl. **117** (1957), 374–377; MR **21** #7439].

W. H. Fleming (Providence, R.I.)

4868:

Miranda, Carlo. Alcune limitazioni integrali per le soluzioni delle equazioni lineari ellittiche del secondo ordine. Ann. Mat. Pura Appl. (4) **49** (1960), 375–384. (English summary)

Let $x=(x_1, \cdots, x_n)$ and let T denote a bounded open set in x-space. Suppose the boundary T^* of T is locally representable by functions $\xi_n=\zeta(\xi_1, \cdots, \xi_{n-1})$ of class $C^{(2)}$. Then there exist a constant K_0 depending only on T, and a constant K_1 depending on T and on a positive parameter λ, such that for every solution $u(x)$ of the elliptic partial differential equation $a_{ik}(x)p_{ik}=f(x)$ in T,

$$\int_T p_{ik}p_{ik}dx \le (4/m_0^2)\int_T f^2dx + K_0\Phi^2 m_1^2/m_0^2 + K_1\int_T u^2dx[(A_\lambda m_1/m_0^2)4\lambda/(\lambda-m) + (A_\lambda m_1/m_0^2)^2 + (m_1/m_0)^2],$$

whenever u is of class $C^{(2)}$, a_{ik} are of class $C^{(1)}$, f is continuous, $0<m_0\eta_i\eta_i\le\eta_i a_{ik}\eta_k\le m_1\eta_i\eta_i$ on T,

$$A_\lambda = [\sum_{i,k,r}\int_T |\partial a_{ik}/\partial x_r|^\lambda dx]^{1/\lambda},$$

$\lambda>n$, $u=\phi$ on T^*, ϕ is of class $C^{(2)}$ on T^*, and

$$\Phi = \Big[\int_T \phi^2 d\sigma + \int_{T^*}\sum\{|\partial\phi/\partial\xi_r|^2 + |\partial^2\phi/\partial\xi_r\partial\xi_s|^2\}\,d\xi_1\cdots d\xi_{n-1}\Big]^{1/2}.$$

A second inequality is given, with different exponents. Finally, extensions of the results are indicated to the case when the solution u, the boundary function ϕ, and the

coefficients a_{ik} are in the completions of their respective spaces in respectively suitable integral norms.

L. M. Graves (Chicago, Ill.)

4869:

Nitsche, Johannes; Nitsche, Joachim. Über reguläre Variationsprobleme. Rend. Circ. Mat. Palermo (2) **8** (1959), 346–353.

The paper concerns solutions $z=z(x, y)$ of elliptic equations $L[z]\equiv a(p, q)r+2b(p, q)s+c(p, q)t=0$ arising from a regular variational problem $\delta\iint F(w)dxdy=0$, where $w=p^2+q^2$, $F'(w)>0$, $1+w\lambda(w)>0$ and $\lambda=2F''(w)/F'(w)$. It is assumed that $L[z]=0$ is very nonlinear in the sense that (1) $\int^\infty w^{-1}[1+w\lambda(w)]\,dw<\infty$. Under these conditions, it is shown that if $z=z(x, y)$ is a C^2-solution of $L[z]=0$ in a punctured disk $0<x^2+y^2\leq R^2$, then $z(0, 0)$ can be defined so that z is a C^2-solution on $x^2+y^2\leq R^2$. This is false if (1) does not hold. It is also shown that if D is a Jordan domain, C its boundary, P_0 a point of C, and if $z(x, y)$ is a C^2-solution of $L[z]=0$ in D satisfying $m\leq\liminf z(x, y)\leq\limsup z(x, y)\leq M$ as $(x, y)\to P\neq P_0$ and $P\in C$, then $m\leq z(x, y)\leq M$ on D. The proof depends on the use of (1) to construct decreasing functions $\Phi(\rho)$, where $\rho^2=x^2+y^2$, satisfying $L[\Phi]\leq 0$ and subject to suitable estimates.

P. Hartman (Baltimore, Md.)

4870:

Suschowk, Dietrich. Lösungen des charakteristischen Anfangswertproblems der Wellengleichung, Tschebyscheffsche Polynome und Legendresche Funktionen 2. Art. Math. Z. **68** (1958), 340–361.

Questions of high-speed aerodynamics lead to the problem of finding a solution $v(t, x, y)$ of the equation (1) $v_{tt}=v_{xx}+v_{yy}$, reducing to zero on the characteristic cone (K): $t^2=x^2+y^2$, $t\geq 0$, together with the derivatives v_t, v_x, v_y, and not identically zero inside this cone: $t>r$, $t>0$, $r=(x^2+y^2)^{1/2}$. It is known, however, that giving a solution of (1) on (K) determines it uniquely everywhere inside (K); accordingly, the "solutions" the author constructs have singularities inside (K). Introducing polar coordinates r, ω, the author puts $v(t, x, y)$ in the form (2) $\sum_{k=0}^\infty v^{(k)}(t, r)(a_k\cos k\omega+b_k\sin k\omega)$, where $v^{(k)}(t, r)$ is a solution of the equation $v_{tt}^{(k)}-v_{rr}^{(k)}-r^{-1}v_r^{(k)}+k^2r^{-2}v^{(k)}=0$ satisfying $v^{(k)}|_{t=r}=v_t^{(k)}|_{t=r}=v_r^{(k)}|_{t=r}=0$, $t>0$, $r>0$, and where the a_k and b_k are constants. It is shown that for the $v^{(k)}$ one may take

$$(3)\quad v^{(k)}(t, r)=\int_1^{t/r}\frac{T_k(\tau)}{\sqrt{(\tau^2-1)}}f_k(t-r\tau)d\tau,\quad t\geq r, r>0,$$

where $f_k(\sigma)$ is an arbitrary function such that $f_k(0)=0$ and $f_k''(\sigma)$ exists and is Riemann integrable on any finite interval, and where $T_k(\tau)$ is the Čebyšev polynomial of degree k. It is established that any finite number of the $v^{(k)}(t, r)$ $(k=0, 1, \cdots)$ are linearly independent for $t\geq r$, $r>0$ (even if they are generated by the same function $f_k(\sigma)\equiv f(\sigma)$). Under the assumption that $f_k(\lambda)=\sum_{n=1}^\infty a_{kn}\lambda^n$, $0\leq\lambda>\Lambda$, there follows from (3) a representation of $v^{(k)}(t, r)$ by a convergent series of Legendre functions of the 2nd kind:

$$v^{(k)}(t, r)=(-1)^k\sum_{n=1}^\infty\frac{n!}{(n+k)!}a_{kn}(t^2-r^2)^{n/2}Q_n^k(t(t^2-r^2)^{-1/2}),$$

$$0\leq t-r<\Lambda, r>0.$$

The nature of the singularities of $v^{(k)}(t, r)$ on the line $r=0$ is found: for a fixed $t=t'>0$, $v^{(k)}(t, r)=O(r^{-k})$, $k>0$, if $\int_0^{t'}(t'-\sigma)^{k-1}f_k(\sigma)d\tau\neq 0$; $v^{(0)}(t', r)=O(\ln r)$ if $f_0(t')\neq 0$. Supplementary restrictions on f_k are found which will ensure convergence of (2).

M. N. Olevskiĭ (RŽMat. **1959** #2691)

4871:

Povzner, A. Ya.; Suharevskiĭ, I. V. Discontinuities of the Green's function of mixed problems for the wave equation. Mat. Sb. (N.S.) **51 (93)** (1960), 3–26. (Russian)

The problem under consideration is the solution of the wave equation $\Delta u-\partial^2u/\partial t^2=0$ in an n-dimensional convex domain D (here detailed for $n=2$) under the conditions $u(0, x)=u_t(0, x)=0$, $u|_S=\phi(t, s)$, $\phi(0, s)=\phi_t(0, s)=0$, where S is the finite differentiable boundary of D. The solution $U(x, k)$ of the equivalent Fourier-transformed problem is obtained as an integral around S of a double-layer potential with density $g(s, k)$ satisfying the integral equation

$$(*)\quad g(x, k)-\frac{ik}{\pi}\int_S g(s, k)\frac{\partial}{\partial n_s}\ln\frac{1}{|x-s|}\,ds\int_0^\infty\frac{e^{ikt}\sigma(t-|x-s|)}{\sqrt{(t^2-|x-s|^2)}}\,dt=2\int_0^\infty e^{ikt}\phi(t, x)\,dt,\quad x\in S,$$

with $\sigma(t)=1$ for $t>0$ and 0 for $t<0$. On setting

$$g_{\mu+1}(x, k)=\int_S H(x, s, k)g_\mu(s, k)\,ds\quad(\mu=0, 1, 2, \cdots),$$

where $H(x, s, k)$ is the coefficient of $g(s, k)$ in the second term of (*), one writes

$$g_{n+1}(s, k)=\int_0^\infty e^{ikt}w_{n+1}(t, s, a)\,dt,\quad s\in S.$$

The function $w(t, x, y)=\sum_{\mu=0}^n w_\mu(t, x, y)+\hat{w}_n(t, s, y)$ and its derivatives have discontinuities at certain points $t=t_j(x, y)$. The function $\hat{w}(t,x,a)$ is a solution of the above boundary-value problem for the wave equation with $\phi(t,s)=w_{n+1}(t,s,a)$. The object of the paper is to find the points $t_j(x, y)$ and the jumps in w and its derivatives with respect to t at these points, for certain classes of domains D.

Theorem 1: For a given $T>0$, a fixed $a\in D$ and any positive integer l, one can find an n such that the function $w_{n+1}(t, s, a)$ is in C_l^0 for $s\in S$, $0\leq t\leq T$. Here a function $f(t, s)$ is in class C_l^0 if it has continuous derivatives $\partial^{k+m}f/\partial t^k\partial s^m$ $(0\leq k+m\leq l)$, and $\partial^j f(0, s)/\partial\nu^j=0$ $(j=0, 1, \cdots, l)$.

Theorem 2: The discontinuities of the function $w(t, x, a)$ and its derivatives with respect to t are found at the points

$$t=t(x, a)=|x-\bar{s}_m|+|\bar{s}_m-\bar{s}_{m-1}|+\cdots+|\bar{s}_1-a|,$$

where $\nabla_\xi(|x-s_m|+\cdots+|s_1-a|)=0$ $(m=1, 2, \cdots)$, and at the point $t=|x-a|$. Here s_j are points of S with coordinates ξ_j, and $\bar{s}_j$ are the associated radius vectors.

R. N. Goss (San Diego, Calif.)

4872:

Maslov, V. P. Application of the methods of functional analysis for construction of the quasi-classical asymptotic

behavior of the solution of the Schrödinger equation. Uspehi Mat. Nauk **14** (1959), no. 3 (87), 161–168. (Russian)

Let $H_\hbar\psi(x) \equiv [-\hbar^2(\partial^2/\partial x^2)+u(x)]\psi(x)$. It is known that the semigroup $A_\hbar(t) \equiv \exp((it/\hbar)H_\hbar)$ gives rise to the solution $\psi_\hbar(x,t) = A_\hbar(t)\psi(x,0)$ of the Schrödinger equation $i\hbar(\partial\psi/\partial t) = H_\hbar\psi$, with Cauchy data $\psi(x,0)$. The author considers only the following special case (which for simplicity we shall here specialize still further, e.g., by taking mass $= \frac{1}{2}$ and energy $= 1$). The variable x is restricted to $a \leq x \leq b$. The function $u(x)$ is smooth, strictly decreasing near a, strictly increasing near b, with $u(a) = u(b) = 1$. Defining $p(x) = (1-u(x))^{1/2}$, the Cauchy data is of the form

$$\psi_\hbar(x,0) = p(x)^{-1/2}\exp[i\hbar^{-1}\int_a^x p(\xi)\,d\xi]\varphi(x),$$

with $\int_a^b |\varphi(x)|^2/p(x)\,dx < \infty$. Henceforth write $L_2'[a,b]$ for the Hilbert space of all such $\varphi(x)$, and $L_2[a,b]$ for the space of ordinary square-integrable functions. Clearly $\varphi(x) \in L_2'$ if and only if $\psi_\hbar(x,0) \in L_2$. The author lets $\hbar \to 0$ and considers the asymptotic behavior in L_2 or L_2' of various entities associated with the Schrödinger equation. Theorem 1: Let $X(x,t)$ be the solution of the Newtonian equation

$$\frac{1}{2}\frac{\partial^2}{\partial t^2}X(x,t) = -u'(X(x,t)),$$

with initial conditions $X(x,0) = x$ and $\partial X(x,0)/\partial t = 2p(x)$. Let $\psi_\hbar(x,0)$ and $\psi_\hbar(x,t)$ be as above. Fix t. Define $F_\hbar(x) = p(x)^{1/2}\psi_\hbar(x,0)$. Then as $\hbar \to 0$, we have

$$p(x)^{1/2}\psi_\hbar(x,t) \sim F_\hbar(X(x,t)),$$

i.e., the difference $\to 0$ in L_2'.

Suppose now that each eigenfunction ψ_k of H vanishes at a and b. But permit now $u(a) < 1$ and $u(b) < 1$. Write $M_\hbar$ for the operator of multiplication by

$$p(x)^{-1/2}\exp[i\hbar^{-1}\int_a^x p(\xi)\,d\xi].$$

Then $M_\hbar$ is isometric from L_2' onto L_2, and transforms an operator A on L_2 into the operator $M_\hbar^{-1}AM_\hbar$ on L_2'. Write P for multiplication by $p(x)$, U for multiplication by $u(x)$, and E for multiplication by 1. Thus $P^2 + U = E$. Theorem 2: Let $\hbar \to 0$. Then $M_\hbar^{-1}(-i\hbar\partial/\partial x)M_\hbar \to P$, hence $M_\hbar^{-1}[(-i\hbar\partial/\partial x)^2 + U]M_\hbar = M_\hbar^{-1}H_\hbar M_\hbar \to E$. Also, $M_\hbar^{-1}((H_\hbar - E)/\hbar)M_\hbar \to -2iP\partial/\partial x$, and

$$M_\hbar^{-1}\exp[it\hbar^{-1}(H_\hbar - E)]M_\hbar \to \exp(-2tP(\partial/\partial x)).$$

The last-named operator can be interpreted as displacement along the trajectories of the associated dynamical system.

Similar asymptotic formulas are exhibited in theorem 3 for the eigenfunction $\psi_\hbar$ of $H_\hbar$ that belongs to the eigenvalue $\lambda_\hbar$ closest to 1. *H. Mirkil* (Hanover, N.H.)

4873:

Martirosyan, R. M. The spectrum of a generalized Schrödinger operator. Akad. Nauk Armyan. SSR. Dokl. **29** (1959), 97–101. (Russian. Armenian summary)

Continuation of the study of the spectra of certain operators of the form $T = -\Delta + S$, where Δ denotes the self-adjoint closure of the Laplacian on L_2 of n-dimensional Euclidean space. Recall that the spectrum of the unperturbed operator $-\Delta$ is purely continuous and consists of the entire positive real axis. The author has shown [Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **10** (1957), no. 1, 85–111; MR **21** #2106] that if $S = M_p$ is multiplication by a function p (Schrödinger operator), where p is bounded, measurable and square-integrable, then the spectrum of T consists of the entire positive real axis along with at most a countable set of eigenvalues, each of finite multiplicity, having no limit points except along the positive real axis. He here gives a new and simplified proof of this result by (1) pointing out, in effect, that the stated conclusion holds for T whenever $B_\lambda S$ is compact for all non-positive λ, B_λ denoting the resolvent of $-\Delta$, and (2) proving that, for p as above, $B_\lambda M_p$ is, in fact, compact for λ non-positive. It follows then that the conclusion continues to hold for any S of the form $S = M_pL + K$ (generalized Schrödinger operator), where L, K are bounded operators with K compact. [Cf. #4851 above.] *A. Brown* (Houston, Tex.)

4874:

Heyda, James F. A note concerning hyperbolic equations with constant coefficients. Quart. Appl. Math. **18** (1960/61), 299–300.

The author gives the formula

$$v(x,y) = v(x_0,y_0)R(x,y;x_0,y_0) + \int_{x_0}^x f(\alpha)R(x,y;\alpha,y_0)\,d\alpha + \int_{y_0}^y g(\beta)R(x,y;x_0,\beta)\,d\beta$$

for the solution of $v_{xy} - \lambda v = 0$ (λ constant) under the conditions $v_x(x,y_0) = f(x)$, $v_y(x_0,y) = g(y)$, where R is the Riemann function $I_0(2[\lambda(x-x_0)(y-y_0)]^{1/2})$.

E. T. Copson (St. Andrews)

4875:

Lees, Milton. Energy inequalities for the solution of differential equations. Trans. Amer. Math. Soc. **94** (1960), 58–73.

Consider the hyperbolic operator

$$Mu(x,t) = \frac{\partial^2 u}{\partial t^2} - a(x,t)\frac{\partial^2 u}{\partial x^2},$$

where $0 < \mu_1 \leq a(x,t) \leq \mu_2$,

$$|a(x,t) - a(x',t')| \leq \mu_0[|x-x'| + |t-t'|],$$

$0 \leq x, x' \leq 1$, $0 \leq t, t' \leq t_0$. It is known [Friedrichs and Lewy, Math. Ann. **98** (1927), 192–204] that for every $u(x,t) \in C^2$, $u(0,t) = u(1,t) = 0$, there exists a constant C, depending only on μ_i ($i = 0, 1, 2$) and t, such that

$$\int_0^1 |u(x,t)|^2\,dx + \int_0^1 \left|\frac{\partial u}{\partial x}(x,t)\right|^2 dx + \int_0^1 \left|\frac{\partial u}{\partial t}(x,t)\right|^2 dx \leq C\left[\int_0^1 \left|\frac{\partial u}{\partial x}(x,0)\right|^2 dx + \int_0^1 \left|\frac{\partial u}{\partial t}(x,0)\right|^2 dx + \int_0^t\int_0^1 |Mu|^2\,dx\,dt\right].$$

In theorem 1, a significant result of the paper, the author proves an analogous inequality for a finite difference operator L which is consistent with M and is defined by

$$L\phi(x,t) = \phi_{\bar{t}\bar{t}}(x,t) - a(x,t)\phi_{\bar{x}x}(x,t),$$
$$(\phi_{\bar{t}}(x,t) = \phi(x,t) - \phi(x,t-k),$$
$$\phi_{\bar{x}}(x,t) = \phi(x+h,t) - \phi(x,t), \text{ etc.}).$$

It follows that L is stable in the mean square norm, without restriction to the mesh ratio h/k.

The author considers also some non-linear cases and semi-discrete approximations of the operator M. In all these cases, corresponding energy inequalities are derived.

In the proofs, some interesting "discrete" lemmas are obtained, as for example the following one. "Let $\omega(t)$, $\rho(t)$ be non-negative functions defined on the discrete set $\Lambda \equiv \{2k, 3k, \cdots, Mk\}$, $k>0$. If $C \geqq 0$, $\rho(t)$ is non-decreasing and $\omega(t) \leqq \rho(t) + Ck \sum_{s=2k}^{t-k} \omega(s)$, then

$$\omega(t) \leqq \rho(t) \exp(C(t-2k))."$$

There are many misprints. *S. Zaidman* (Bucharest)

4876:

Berezanskiĭ, Yu. M. On an operator derived from an ultrahyperbolic differential expression. Ukrain. Mat. Ž. **11** (1959), 315–321. (Russian)

Let G be either a bounded domain in the $(n'+n'')$-dimensional real space with the boundary composed only of portions of hyperplanes parallel to the coordinate hyperplanes, or a domain of the form $G' \times G''$, where G' [G''] is a bounded domain in the n' [n''] dimensional real space with sufficiently regular boundary. The author proves that the closed operator L generated by

$$\mathscr{L}[u] = -\sum_{i=1}^{n'} \frac{\partial^2 u}{\partial x_i^2} + \sum_{j=1}^{n''} \frac{\partial^2 u}{\partial y_j^2} + c(x)u$$

(where $c(x)$ is a bounded real function) defined in a suitable manner in $L^2(G)$, is self-adjoint. In the second case the two laplacian operators in $\mathscr{L}[u]$ can be replaced by two formally self-adjoint elliptic differential operators (of order 2). In the case $n'=n''=1$, $G=[0, l]\times[0, k]$, the following interesting result (concerning the first homogeneous boundary problem for $-\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = f$) is proved: the spectrum of L does not contain 0 if and only if $\alpha = k/l$ is irrational and in its development in continued fractions, $\alpha = a_0 + 1/a_1 + \cdots$, the sequence $(a_0, a_1, \cdots)$ is bounded. *C. Foiaş* (Bucharest)

4877:

Krivenkov, Yu. P. Representations of solutions of the Euler-Poisson-Darboux equation with negative coefficient. Dokl. Akad. Nauk SSSR **123** (1958), 239–242. (Russian)

Integral representations of solutions of the equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{c}{y}\frac{\partial w}{\partial y} = 0 \quad (c = \text{const})$$

in terms of analytic functions have already been given by the author for the case of $c>0$ [same Dokl. **116** (1957), 351–354, 545–548; MR **20** #4076, 4077]. The present note extends these results to the case of negative c.

M. G. Arsove (Seattle, Wash.)

4878:

Krivenkov, Yu. P. Some problems on the Euler-Poisson-Darboux equation. Dokl. Akad. Nauk SSSR **123** (1958), 397–400. (Russian)

Let T be a region in the upper-half-plane, and let L be a segment of the real axis forming part of the boundary of T. Relative to the above equation, for w a function C'' on T and continuous on its closure, the author considers the boundary problems determined by functions f_1 and f_2 on L as follows: (1) $w(x, 0) = f_1(x)$ $(x \in L)$, $c \geqq 1$; (2) $w(x, 0) = f_1(x)$ and $\lim_{y\to 0} y^c \partial w/\partial y = f_2(x)$ $(x \in L)$, $0 < |c| < 1$. (A third boundary problem analogous to (2), but for the region obtained by symmetrization of T about the x-axis, is also considered.) The solutions of these boundary problems are known to be unique. Criteria for the existence of solutions are presented here in terms of the existence of certain analytic functions, the methods being based on the integral representations noted in the preceding paper.

M. G. Arsove (Seattle, Wash.)

4879:

Szybiak, A. On the asymptotic behaviour of the solutions of the equation $\Delta u - \partial u/\partial t + c(x)u = 0$. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 183–186. (Russian summary, unbound insert)

Consider the equation $\Delta u - \partial u/\partial t + c(x)u = 0$, where $\Delta = \sum_{i=1}^{m} \partial^2/\partial x^{i2}$, $x = (x^1, \cdots, x^m)$, and $c(x)$ is a bounded non-negative function which satisfies a Hölder condition in m-dimensional cartesian space E_m. Theorem 1: If $c(x) \geqq 0$ in E_m and $c(x) \not\equiv 0$, then the corresponding fundamental solution $u(t, x, s, y)$ tends to $+\infty$ as $t \to \infty$, and the convergence is of exponential order and uniform in x and y on every bounded set. Theorem 2: If $u_s(t, x)$ is the solution with the initial condition $u_s(s, x) = \varphi(x)$, where $\varphi(x) \geqq 0$ and $\varphi(x) \not\equiv 0$, then $u_s(t, x)$ tends to $+\infty$ as $t \to +\infty$, the convergence being of exponential order and uniform on every bounded set. The analysis is based on the formula for the fundamental solution: $u(t, x, s, y) = \sum_{n=0}^{\infty} u_n(t, x, s, y)$, where

$$u_0(t, x, s, y) = (4\pi(t-s))^{-m/2} \exp\{-(4(t-s))^{-1}|x-y|^2\},$$

$$u_n(t, x, s, y) = \int_s^t d\tau \int u_0(t, x, \tau, \xi) C(\xi) u_{n-1}(\tau, \xi, s, y) d\xi \qquad (n = 1, 2, \cdots),$$

which was given by F. G. Dressel [Duke Math. J. **7** (1940), 186–203; **13** (1946), 61–70; MR **2**, 204; **7**, 450].

J. B. Diaz (College Park, Md.)

4880:

Ladyženskaya, O. A. Solution of the first boundary problem in the large for quasi-linear parabolic equations. Trudy Moskov. Mat. Obšč. **7** (1958), 149–177. (Russian)

Consider the boundary problem

$$Lu \equiv u_{,t} - \sum_{i,j=1}^{n} a_{ij}(x, t, u) u_{,ij} + \sum_{i=1}^{n} a_i(x, t, u) u_{,i} + a(x, t, u) = 0, \tag{1}$$

$$u|_{t=0} = \phi(x), \quad u|_{\partial\Omega} = 0,$$

for $(x, t) \in \Omega \times [0, T] = Q$, where Ω is a closed bounded region in E^n. Assume that L is uniformly parabolic (with modulus α) and $a_{,u}(x, t, u) \geqq \beta$ for $(x, t) \in Q$ and all u. The author shows that these conditions imply the existence of a constant C such that every solution of (1) satisfies $|u| \leqq C$. It is further assumed that $\max |a_{ij,u}| \leqq \alpha e 3^{1/2}/12nC$ for $(x, t, u) \in Q \times [-C, C]$. In the first part of the paper the author derives a priori estimates for a solution of (1) and its derivatives. In the second part the author proves that under certain regularity assumptions on the data and coefficients of L there exists a unique generalized solution (i.e., with $u_{,t}$ and $u_{,ij} \in L_2(Q)$) of (1). The proof is based on the method of Rothe [Math. Ann. **102** (1930),

650–670] and the analogues of the estimates derived in the first part for the resulting difference-differential equation problem. Moreover, the author proves that her generalized solution is a classical solution provided certain additional regularity conditions are satisfied. These results were announced in Dokl. Akad. Nauk SSSR **107** (1956), 636–639 [MR **17**, 1214].

D. G. Aronson (Minneapolis, Minn.)

4881:

Chen, Y. W. Hoelder continuity and initial value problems of mixed type differential equations. Comment. Math. Helv. **33** (1959), 296–321.

Study of the behavior of solutions of the Tricomi equation $x^m U_{yy} + U_{xx} = 0$ as the parabolic line $x = 0$ is approached from the elliptic region $x > 0$. Hölder continuity properties of solutions of equations of mixed type at the parabolic line, and, in particular, a generalization of Privaloff's theorem. These results, together with a reflection law for Tricomi's equation, are derived by continuing the solutions concerned analytically into the complex domain. *P. R. Garabedian* (New York)

4882:

Lions, J. L. Opérateurs de transmutation singuliers et équations d'Euler Poisson Darboux généralisées. Rend. Sem. Mat. Fis. Milano **28** (1959), 124–137.

Lecture on the notion of transmutation introduced by Delsarte in 1938. Let $\mathscr{E}_*$ be the space of all even complex infinitely differentiable functions on the real line R_x. The linear differential operator $f \to M_p f = D^2 f + x^{-1} M(x) Df + N(x) f$ (where $D = d/dx$, $M(x), N(x) \in \mathscr{E}_*$, and we write $M(0) = 2p + 1$ for convenience) belongs to the vector space $\mathscr{L}(\mathscr{E}_*; \mathscr{E}_*)$ of all continuous linear transformations of $\mathscr{E}_*$ into itself. A transmutation operator of M_p into D^2 is an isomorphism X_p of $\mathscr{E}_*$ onto itself such that $D^2 X_p = X_p M_p$. Write $X_p^{-1} = \mathfrak{X}_p$. After recalling the particular case $M(x) =$ constant and $N(x) = 0$, in which a classical Sonine operator is such a X_p and a classical Poisson operator is such a $\mathfrak{X}_p$, the author indicates that, in the general case, there is one and only one entire function $p \to X_p$ with values in $\mathscr{L}(\mathscr{E}_*; \mathscr{E}_*)$ such that X_p is a transmutation operator of M_p into D^2, except for $p = -1, -2, \cdots$, such that $X_p(f(0)) = f(0)$ for $f \in \mathscr{E}_*$. This is called a singular transmutation due to the presence in M_p of a singularity at the origin. $\mathfrak{X}_p$ is defined by a kernel in the sense of L. Schwartz which is a distribution $\mathfrak{X}_p(x, y)$ whose support is contained in the set $|y| \leq |x|$. We have

$$\mathfrak{X}_p(x, y) = 1/\pi \int_0^\infty \theta(x, s) \cos(sy)\, ds$$

in the distribution sense, where the infinitely differentiable function $\theta(x, s) = \mathfrak{X}_p(\cos(sx))$ is determined by the ordinary equation $M_p\theta(x, s) + s^2\theta(x, s) = 0$, $\theta(0, s) = 1$, θ even in x. On the other hand, the singular Cauchy problem

$$(M_p)_x v(x, y) - D_y^2 v(x, y) = 0,$$
$$v(x, 0) = f(x), \quad D_y v(x, 0) = 0$$

has one and only one infinitely differentiable solution $v(x, y)$ which is even in x and y, for $p \neq -1, -2, \cdots$, and then $X_p f(y) = v(0, y)$, where f is infinitely differentiable. The following application to the Euler-Poisson-Darboux equation is indicated. Let $A = -\sum_{i,j=1}^n D_i(a_{ij}(x) D_j)$ in R_x^n, where $a_{ij} = a_{ji}$ are real infinitely differentiable functions and A is uniformly elliptic, meaning that $\sum a_{ij}(x)\xi_i\xi_j \geq \alpha \sum |\xi_i|^2$ with $\alpha > 0$. Then the singular Cauchy problem

$$A_x u(x, t) + (M_p)_t u(x, t) = 0,$$
$$u(x, 0) = f(x), \quad D_t u(x, 0) = 0,$$

has one and only one infinitely differentiable solution $u(x, t)$ even in t, f being infinitely differentiable, and $u(x, t) = (\mathfrak{X}_p)_t w(x, t)$, where $w(x, t)$ is the unique infinitely differentiable solution even in t of the usual Cauchy problem

$$A_x w(x, t) + D_t^2 w(x, t) = 0,$$
$$w(x, 0) = f(x), \quad D_t w(x, 0) = 0,$$

provided $p \neq -1, -2, \cdots$ holds. Several remarks concerning further extensions and applications to fractional integration, Hankel transform, and mean-periodic functions, and a bibliography are added.

L. Nachbin (Waltham, Mass.)

4883:

Collatz, Lothar. Application of the theory of monotonic operators to boundary value problems. Boundary problems in differential equations, pp. 35–45. Univ. of Wisconsin Press, Madison, 1960.

This paper is expository. It gives a number of examples illustrating procedures for obtaining existence proofs or estimates for solutions of equations $Tu = u$, where u is an element of a Banach space R, T is a map of R into itself, there is an order relation $u \leqq v$ in R, and T is "monotone" in some sense. *P. Hartman* (Baltimore, Md.)

4884:

Mihaĭlov, L. G. An investigation of the generalized Cauchy-Riemann system where the coefficients have first order singularities. Dokl. Akad. Nauk SSSR **129** (1959), 507–510. (Russian)

I. N. Vekua [Mat. Sb. (N.S.) **31** (**73**) (1952), 217–314; MR **15**, 230] studied the generalized Cauchy-Riemann system

$$\frac{\partial w}{\partial \bar{z}} = A(z)w + B(z)\overline{w} + F(z)$$

with continuous A, B, F; and later with A, B, F in L_p, $p > 2$ [Czechoslovak Math. J. **6** (**81**) (1956), 143–160; MR **19**, 308]. The author studies the singular system

$$\frac{\partial w}{\partial \bar{z}} = \frac{a(z)}{|z|} w + \frac{b(z)}{|z|} \overline{w} + \frac{c(z)}{|z|};$$

its corresponding integral equation

$$w(z) = g(z) - \frac{1}{\pi} \iint_D \frac{a(\zeta)w(\zeta) + b(\zeta)\overline{w(\zeta)}}{|\zeta|(\zeta - z)}\, ds,$$

where

$$g(z) = \Phi(z) - \frac{1}{\pi} \iint_D \frac{c(\zeta)}{|\zeta|(\zeta - z)}\, ds,$$

with $\Phi(z)$ holomorphic; and the fundamental boundary value problems for the singular elliptic equation

$$\Delta u + \frac{p(x, y)}{r} u_x + \frac{q(x, y)}{r} u_y = \frac{f(x, y)}{r},$$

$r = \sqrt{(x^2 + y^2)}$; seeking to extend Vekua's results for the regular system. *J. B. Diaz* (College Park, Md.)

4885:

Šmidt, V. [Schmidt, Wolfgang.] A generalized Riemann-Hilbert problem in the case of a negative index. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 893–895. (Russian)

The author considers the boundary value problem $\mathrm{Re}(t^{-n}U(t))=\gamma(t)$, $|t|=1$, for solutions U of the differential equation $U_z-A\overline{U}=0$ in the unit disc $|z|<1$, with $n<0$. The general Riemann-Hilbert problem with negative index may always be reduced to this form. The author observes that the boundary value problem always has a solution, if U is permitted to have a pole at $z=0$ of order at most n. Using a representation formula for such generalized solutions, he obtains necessary and sufficient conditions on $\gamma(t)$ for the existence of regular solutions. By comparing these conditions with those found by I. N. Vekua [Mat. Sb. (N.S.) **31 (73)** (1952), 217–314; MR **15**, 230], he obtains a representation of the solutions w of the homogeneous adjoint problem $\mathrm{Re}\{t^n t'(s)w(t)\}=0$ (s is arc length on $|t|=1$), $w_{\bar{z}}+\overline{A}\overline{w}=0$, in terms of a solution of the original problem.

W. Koppelman (New Haven, Conn.)

4886:

Trèves, François. Opérateurs différentiels hypoelliptiques. Ann. Inst. Fourier. Grenoble **9** (1959), 1–73.

This article continues and completes previous work by Hörmander, Malgrange and Mizohata on hypoellipticity of linear partial differential equations with variable coefficients. Chapter I presents preliminary results on some spaces of distributions. The essential results of chapter II are as follows. Let V be an infinitely differentiable connected manifold and $v\to P(v, D_x)$ an infinitely differentiable family, where $v\in V$ and each $P(v, D_x)$ is a linear differential operator with constant coefficients in $R_x{}^n$ of order $m\geq 1$, the coefficients of $P(v, D_x)$ being then infinitely differentiable functions on V. Theorem 2.1: Assume that this family $v\to P(v, D_x)$ is formally hypoelliptic, that is, each $P(v, D_x)$ is hypoelliptic on $R_x{}^n$ and two $P(v_1, D_x)$, $P(v_2, D_x)$ are equally strong. Then there are strictly positive numbers s, d and an infinitely differentiable mapping $v\to E(x, v)$ of V into the space of distributions $A^{s,d}_{\mathrm{loc}}$ such that, for each $v\in V$, $E(x, v)$ is an elementary solution, that is, $P(v, D_x)E(x, v)=\delta_x$. As a converse, the following result is established. Theorem 2.2: Suppose that (TA) $v\to P(v, D_x)$ is of analytic type, in the sense that every linear combination of the coefficients of $P(v, D_x)$ considered as functions of v on V, having a zero of infinite order at a point of V must be identically zero on V. Suppose further that (P) there is an infinitely differentiable function $v\to E(x, v)$ on V with values in the space of distributions $\mathscr{D}_x'$ such that (I) there is a real number s such that the convolution operator (in x) $E(x, v)*$ is an infinitely differentiable function of v in V with values in the space $L_b(H_s^0, H^0_{\mathrm{loc}})$ of continuous linear transformations between the spaces of distributions therein; (II) for every function $\alpha(x)\in\mathscr{D}_x$ having its support away from the origin, $\alpha(x)E(x, v)$ is an infinitely differentiable function of v in V with values in $\mathscr{D}_x$; (III) for every v in V, $P(v, D_x)E(x, v)=\delta_x$. Then there results that the family $v\to P(v, D_x)$ is formally hypoelliptic. A simple example of a differential operator of second order on the real line depending on a real parameter shows that this converse breaks down if (TA) is not assumed. Chapter III establishes (theorem 3.2) a sufficient condition for a linear differential operator $P(x, D_x)$ with infinitely differentiable coefficients in an open set Ω of $R_x{}^n$ to be hypoelliptic. This condition subsumes the Hörmander-Malgrange criterion, according to which $P(x, D_x)$ is hypoelliptic if the family $v\to P(v, D_x)$ for $v\in\Omega$ is formally hypoelliptic. An example is also provided of an hypoelliptic equation of fourth order which fails to satisfy the Hörmander-Malgrange criterion, no matter what change of variables is made.

L. Nachbin (Waltham, Mass.)

POTENTIAL THEORY

4887:

✓ ★**Séminaire de Théorie du Potentiel, dirigé par M. Brelot, G. Choquet et J. Deny; 2e année: 1958.** Faculté des Sciences de Paris. Secrétariat mathématique, Paris, 1959. ii+71 pp. (mimeographed)

In the first exposition, 40 pages long, M. Brelot presents in detail, with some complements and a few corrections, the contents of three notes [C. R. Acad. Sci. Paris **245** (1957), 1688–1690; **246** (1958), 2334–2337, 2709–2712; MR **21** #5094a, #5094b, #5095 {in the next to last paragraph of 5094b the term "$\overline{p}$-function" should be changed to "p-function" both in Axiom III′ and in the definition of potential, while in the 14th line of #5095 the term "p-function" should be changed to "$\overline{p}$-function"}]. The space Ω is now assumed to be locally connected as well as connected, locally compact and not compact; the p-, $\overline{p}$-, $\overline{p}^*$-functions are now called respectively harmonic, hyperharmonic and superharmonic. If $\mathscr{B}$ is a collection of regular domains forming a basis for the topology of Ω, then a function v defined on an open set ω_0 is said to be an $\mathscr{S}_{\mathscr{B}}$-function on ω_0 if it is locally bounded below and if $v(x)\geq\int v d\rho_x{}^\omega$ for $x\in\omega\in\mathscr{B}$, $\omega\subset\omega_0$; these functions are useful in proving theorems concerning hyperharmonic functions.

Part I, pp. 1-01 to 1-17, develops the consequences of Axioms I, II, III [MR **21** #5094a]. After establishing the elementary properties of hyperharmonic and $\mathscr{S}_{\mathscr{B}}$-functions, the author proves, among other results, two theorems concerning an open set ω_0 on which there exists a strictly positive potential. Theorem 5: There exists on ω_0 a strictly positive, finite, continuous potential. Theorem 6 [Mme. Hervé, see also ibid. **247** (1958), 1956–1959; MR **21** #5907]: Every finite continuous function on a compact subset of ω_0 can be uniformly approximated by a difference of finite continuous potentials on ω_0. The second theorem implies that, if a strictly positive potential exists on Ω and if ω is an open set with compact closure, then every continuous function on the boundary of ω is resolutive. The last three pages of this part are devoted to determining sets; a regular set ω is 'completely determining' [respectively, 'determining'] if, for v_1 and v_2 nonnegative superharmonic on Ω [respectively, nonnegative superharmonic and locally bounded] and harmonic on ω, the equality $v_1=v_2$ on $\Omega-\omega$ implies equality on all of Ω. The following is typical of the technically useful theorems proved. Theorem 9: Consider on Ω a decreasing directed family of nonnegative $\mathscr{S}_{\mathscr{B}}$-functions v_t and a regular determining set ω belonging to $\mathscr{B}$; then $\int \inf v_t d\rho_x{}^\omega=\inf\int v_t d\rho_x{}^\omega$.

Part III, pp. 1-17 to 1-31, treats the Riesz-Martin

representation of harmonic and superharmonic functions [see MR **21** #5094b], using theorems of G. Choquet [ibid. **243** (1956); 555–557, 699–702; MR **18**, 288, 219]. The nonnegative superharmonic functions are shown to form a lattice, under Axioms I, II, III. Further axioms are introduced in order to secure the compactness needed in applying the theorems of Choquet. For the representation of nonnegative harmonic functions, it suffices to impose in addition to Axioms I and II the condition that Ω be the countable union of compact sets and the Axiom III′: For each domain δ, a nonnegative harmonic function on δ either vanishes identically or is strictly positive; for each domain δ and each $x_0 \in \delta$, the nonnegative harmonic functions on δ with value 1 at x_0 are equicontinuous at x_0. For the representation of nonnegative superharmonic functions, Brelot imposes in addition to Axioms I, II, III′ the condition that Ω satisfy the second axion of countability and the Axiom IV: The regular completely determining sets form a base for the topology of Ω. (Axioms III′ and IV are a little stronger than the corresponding ones in the note reviewed in MR **21** #5094b.)

Part III, pp. 1-31 to 1-40, is devoted to polar sets and convergence theorems; see MR **21** #5095.

The second exposition merely states a theorem of the reviewer.

In the third exposition, 30 pages long, G. Anger discusses another axiomatization of potential theory, somewhat in the spirit of a note by G. Choquet and J. Deny [ibid. **243** (1956), 764–767; MR **19**, 848]. He takes as basis a locally compact space E and a nonnegative kernel Φ defined on $E \times E$. A positive measure λ is said to be in F^+ if S_λ (the support of λ) is compact and if the function $T'\lambda \equiv \int \Phi(x, \cdot)\lambda(dx)$ is continuous on E; a positive measure μ is said to be in G^+ if $\int T'\lambda \, d\mu$ is finite for all λ in F^+; F and G are the corresponding linear spaces of differences, while D is the linear space of functions $T'\lambda$, with λ in F. The author's aim is to develop potential theory by applying to F, D, G methods from the theory of topological vector spaces supported by certain of the following axioms. (A_1) $F^+ \neq (0)$. (A_2) Let B be a (universally measurable) subset of E, let $\lambda \in F^+$, let $\lambda(B) > 0$; then there is λ_0 in F^+ such that $S_{\lambda_0} \subset B$ and $\lambda_0(B) > 0$. (A_3) The spaces D and G are in duality under the bilinear form $(T'\lambda, \mu) = \int T'\lambda \, d\mu$. ($A_4$) For $\lambda \in F$, $T'\lambda(x) \to 0$ as x approaches the Alexandroff point of E. (A_5) (Stated for a given compact set K.) For each $a > 0$ there is a λ in F^+ such that $S_\lambda \subset K$ and $T'\lambda(x) \geq a$ for $x \in K$. In pp. 3-01 to 3-06 the author reviews the theory of the Newtonian potential, showing in particular that the axioms hold (with K a set of strictly positive capacity in (A_5)). In pp. 3-06 to 3-11, (A_1)–(A_4) are stated and a number of elementary consequences are deduced. Various convergence theorems for potentials (functions $T\mu = \int \Phi(\cdot, y) \, d\mu(z)$, with μ in G) are discussed in pp. 3-11 to 3-15. The last 15 pages treat several "principles of balayage," with (A_2) strengthened slightly and the emphasis placed on equivalent restatements of the principles. The author has discussed symmetric kernels in another paper [Wiss. Z. Tech. Hochsch. Dresden. **8** (1958/59), 679–685; MR **22** #846].

G. A. Hunt (Princeton, N.J.)

4888:

Ėidus, D. M. Inequalities for Green's functions. Mat. Sb. (N.S.) **45 (87)** (1958), 455–470. (Russian)

Let Ω be a region in 3-space bounded by a Lyapunov surface S of index λ [the terminology here seems to be that of N. M. Gunther, *La théorie du potential et ses applications aux problèmes fondamentaux de la physique mathématique*, Gauthier-Villars, Paris, 1934]. Green's function for Ω has the form $G(x, y) = (4\pi r_{xy})^{-1} + g(x, y)$, where r_{xy} is the distance between x and y and g is the harmonic part of G. The present article is concerned with generalizations of the inequality $|Dg(x, y)| \leq Cr_{xy}^{-2}$, in which $Dg(x, y)$ signifies the first order partial derivative of g with respect to any of the coordinates of x, and C is a constant depending only on Ω. It is shown that g here can be replaced by the function

$$g_1(x, y) = \frac{1}{8\pi^2} \int_S \frac{1}{r_{xt}} \frac{\partial}{\partial n_t} \left(\frac{1}{r_{ty}}\right) dS_t,$$

which is, in a sense, the smooth part of g. Corresponding inequalities are given for $|g_1|$, $|g - g_1|$, $|D(g - g_1)|$, $|D^2 g_1|$, and $|D^2(g - g_1)|$.

M. G. Arsove (Seattle, Wash.)

4889:

Ejdusz, D. M. [Ėidus, D. M.]. Inequalities for Green's functions. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **9** (1959), 315–332. (Hungarian)

Translation of the paper reviewed above.

4890:

Curtiss, J. H. Interpolation with harmonic and complex polynomials to boundary values. J. Math. Mech. **9** (1960), 167–192.

The theory of a possible method for getting polynomial approximations to the solution of the Dirichlet problem by interpolation to the boundary data is investigated. An analogous but more accessible problem for complex polynomial approximation is examined in some detail to serve as basis for conjectures. Then surmises concerning the harmonic problem, with respect to the existence of "universal" sequences $\{S_n\}$: $\{z_{1n}, z_{2n}, \cdots, z_{nn}\}$, $n = 1, 2, \cdots$, of preassigned sets (z_{kn}) of interpolation on the curve C, the boundary of the region R in the z plane $(z = x + iy)$, are stated and discussed.

The paper explains, in § 2, known results on the uniqueness of complex and of harmonic polynomials found by interpolation at $\{S_n\}$; in §§ 3–5, the convergence theory for complex polynomials, including sufficient conditions for $\{S_n\}$ in the case when the function $f(z)$ that is approximated is analytic in R and continuous on $R + C$ (theorem 4); in § 6, simultaneous approximations on two or more mutually exterior regions; and, in §§ 7, 8, interpolation by harmonic polynomials on a general Jordan curve and especially on the unit circle. Also a number of new results are deduced. Theorem 2, for instance, states that, if C is rectifiable and contains $z = 0$ in its interior, and $f(t)$ is defined and continuous on C, and if

$$\omega_n(z) = \prod_{k=1}^{n} (z - z_{kn}), \quad \lambda_k(z) = \frac{\omega_n(z)}{\omega_n'(z_{kn})(z - z_{kn})},$$

$$L_n(g; z) = \sum_{k=1}^{n} g(z_{kn}) \lambda_k(z),$$

then the Lagrange polynomials $L_n(f; z)$ converge at a

single given point z_0 in R to the function

$$F(z) = (2\pi i)^{-1} \int_C (t-z)^{-1} f(t)\, dt$$

if and only if (4.6) (a) $\lim_{n\to\infty} L_n(z^{-k}; z_0) = 0$, (b) $\sum_1^n |\lambda_k(z_0)| < M$ $(k = 1, 2, \cdots, n;\ n = 1, 2, \cdots;\ U$ independent of $n)$; the convergence of $\{L_n(f; z)\}$ to $F(z)$ is uniform on any closed point set of R on which (4.6) holds uniformly. *H. Kober* (Birmingham)

4891:

Walsh, J. L. Solution of the Dirichlet problem for the ellipse by interpolating harmonic polynomials. J. Math. Mech. **9** (1960), 193–196.

The following result is proved. Theorem: Let the boundary B of a bounded region D in the (u, v) plane be an ellipse, and let the function $U(u, v)$ be defined and satisfy a Lipschitz condition of some positive order on B with respect to arc length. Let the harmonic polynomial $p_n(u, v)$ of degree not greater than n coincide with $U(u, v)$ in $2n+1$ points equally spaced on B with respect to the conjugate of Green's function for the exterior of $D+B$ with pole at infinity. Then the sequence $p_n(u, v)$ converges to $U(u, v)$ uniformly on B, hence converges uniformly in $D+B$ to the corresponding solution of the Dirichlet problem.

The corresponding problem for a circular disc was dealt with before by Walsh [Bull. Amer. Math. Soc. **35** (1929), 499–544], where also the general problem when B is a Jordan curve was formulated. The present proof is based on the conformal representation $w = \frac{1}{2}(z+z^{-1})$ $(w = u+iv,\ z = x+iy = re^{i\theta})$ which maps the interior D of the two-sheeted ellipse $|w-1|+|w+1| = R+1/R$ $(R>1)$, suitably slit, to the ring $1/R < |z| < R$, the harmonic polynomials $p_n(u, v)$ of degree $\leqq n$ into trigonometric polynomials

$$\pi_n(z) = \sum_{k=0}^{n} A_{n,k}(r^k + r^{-k}) \cos k\theta + \sum_{k=1}^{n} (r^k - r^{-k}) B_{n,k} \sin k\theta \quad (r > 0),$$

and the preassigned points on B to equally spaced points on $|z| = R$ and $|z| = 1/R$. Using a result due to D. Jackson, the author deduces the uniform convergence of the $\{\pi_n\}$ on $|z| = R$ and, therefore, that of the $p_n(u, v)$ to $U(u, v)$ on $D+B$ and so to a harmonic function inside the ellipse. *H. Kober* (Birmingham)

4892:

Kalik, Carol. La solution du problème de Neumann dans le domaine multiplement connexe, limité par un nombre fini de sphères. Com. Acad. R. P. Romîne **8** (1958), 745–753. (Romanian. Russian and French summaries)

G. M. Goluzin [Mat. Sb. **41** (1934), 277–283] has given a method, which generalizes the classical "alternating" procedure of H. A. Schwarz for the solution of the classical Dirichlet problem, for the Dirichlet problem for domains bounded by a finite number of spheres. The author deals with the Neumann problem. For definiteness, domains bounded by three spheres are considered.

J. B. Diaz (College Park, Md.)

4893:

Gehring, F. W. Harmonic functions and Tauberian theorems. Proc. London Math. Soc. (3) **10** (1960), 88–106.

Let H^B, H^+, and H denote the classes of functions $u(z)$ that are harmonic in the upper half-plane and satisfy the respective conditions: (i) $u(z)$ is bounded; (ii) $u(z)$ is bounded below; (iii) $u(z) = u_1(z) - u_2(z)$ $(u_1, u_2 \in H^+)$. For each u in H, there exists an essentially unique function $\alpha(t) = \alpha(t; u)$ such that

$$u(x+iy) = Cy + (y/\pi) \int_{-\infty}^{\infty} [(x-s)^2 + y^2]^{-1} d\alpha(s).$$

Some relations have been established between the existence of the segmental limit $U(\theta) = \lim_{r\to 0+} u(re^{i\theta})$ and the existence of the "weighted symmetric derivative" $A(\phi) = \lim_{t\to 0+} [(\pi-\phi)\alpha(t) - \phi\alpha(-t)]/\pi t$. Indeed, if $u \in H^+$, then the existence of $U(\pi/2)$ is necessary and sufficient for the existence of $A(\pi/2)$; on the other hand Allen and Kerr [J. London Math. Soc. **29** (1954), 104–115; MR **15**, 526] have shown that for $\theta = \phi \neq \pi/2$ the existence of $U(\theta)$ is neither necessary nor sufficient for the existence of $A(\phi)$, even in H^B. The author of the present paper pursues the problem further and obtains the following definitive solutions.

(1) If $0 < \phi < \pi$ and $\phi \neq \pi/2$, then there exists $u \in H^B$ such that $(\pi-\phi)\alpha(t) - \phi\alpha(-t) = 0$ for all $t > 0$, and such that $U(\theta)$ does not exist for any θ in $(0, \pi)$.

(2) If $0 < \theta < \pi$ and $\theta \neq \pi/2$, then there exists $u \in H^B$ such that $u(re^{i\theta}) = 0$ for all $r > 0$, and such that $A(\phi)$ does not exist for any ϕ. (Note: if $u \in H^+$ and $U(\theta)$ exists for two values of θ, then $\alpha(t)$ has left- and right-hand derivatives at $t = 0$, and $U(\theta)$ exists for all θ.)

(3) If $u \in H^+$ and $0 < \theta < \pi$, and if $U(\theta)$ and $A(\theta)$ both exist, they are equal.

(4) Let $u \in H^+$, $0 < a < \pi$, $0 < b < \pi$. Suppose that $b < \pi/2$ and $a > b$, or $b = \pi/2 \neq a$, or $b > \pi/2$ and $a < b$. If $A(a)$ and $U(b)$ exist, then, for each θ in $(0, \pi)$, $A(\theta)$ and $U(\theta)$ exist and are equal.

(5) Let $0 < a < \pi$, $0 < b < \pi$. Suppose that $b < \pi/2$ and $a < b$, or $a = \pi/2 = b$, or $b > \pi/2$ and $a > b$. Then there exists $u \in H^B$ such that $(\pi-a)\alpha(t) - a\alpha(t) = 0 = u(te^{ib})$ for all $t > 0$, and such that $A(\phi)$ and $U(\theta)$ do not exist except for $\phi = a$ and $\theta = b$, respectively.

(6) If $0 < a < \pi$, $a \neq \pi/2$, then there exists $u \in H^B$ such that $(\pi-a)\alpha(t) - a\alpha(-t) = 0 = U(a)$ for all $t > 0$, and such that $A(\phi)$ and $U(\theta)$ do not exist except for $\phi = a$ and $\theta = a$.

(7) If $0 < a < \pi$, $a \neq \pi/2$, $u \in H^+$, and $(\pi-a)\alpha(t) - a\alpha(-t) = 0 = u(te^{ia})$ for all t, then there exists a constant C such that $u(te^{i\theta}) = C(\theta - a)$ for $0 < t < \infty$, $0 < \theta < \pi$.

Those proofs which require the construction of a specific example are based largely on the function

$$\alpha(t) = \sinh a\mu \Re\left(\frac{t^{i\mu+1}}{i\mu+1}\right), \quad \alpha(-t) = \sinh(\pi-a)\mu \Re\left(\frac{t^{i\mu+1}}{i\mu+1}\right) \quad (t > 0),$$

where μ denotes an appropriate positive constant. The other arguments are based on a convexity principle and a Tauberian theorem due to Beurling.

G. Piranian (Ann Arbor, Mich.)

4894:

Górski, Jerzy. The method of extremal points and Dirichlet's problem in the space of two complex variables. Arch. Rational Mech. Anal. **4**, 412–427 (1960).

Bergman has set the problem of finding a natural class of real functions of several complex variables whose real parts are dense in the space of continuous functions on the Bergman-Šilov boundary. Bremermann has shown that the plurisubharmonic functions have this property for certain domains. The author shows how a method of solving Dirichlet's problem in the plane due to Leja produces a plurisubharmonic function having desired boundary values for some domains.

D. B. Lowdenslager (Princeton, N.J.)

4895:

Džafarov, A. S. Some properties of n-harmonic functions. Dokl. Akad. Nauk SSSR **128** (1959), 454–455. (Russian)

An n-harmonic function is defined as a function $u(\rho_1, \vartheta_1; \cdots; \rho_n, \vartheta_n)$ which is harmonic in each of the variable pairs (ρ_k, ϑ_k) $(k=1, 2, \cdots, n)$. Let u be an n-harmonic function on the cylinder set

$$D = \{(\rho_k, \vartheta_k): 0 \leqq \rho_k < 1, 0 \leqq \vartheta_k \leqq 2\pi\ (k = 1, 2, \cdots, n)\},$$

and let $f(\vartheta_1, \cdots, \vartheta_n)$ be a function defined for $0 \leqq \vartheta_k \leqq 2\pi$ $(k=1, 2, \cdots, n)$. Suppose further that u takes on the boundary values given by f, in the sense that u is continuous on $\bar{D}$ and satisfies

$$u(1, \vartheta_1; \cdots; 1, \vartheta_n) = f(\vartheta_1, \cdots, \vartheta_n).$$

For f and certain of its partial derivatives assumed to belong to L_p, the author gives a number of rather complicated inequalities on the higher order finite differences of u taken in the direction of a fixed coordinate axis x_k.

M. G. Arsove (Seattle, Wash.)

4896:

Panič, O. I. Potentials for polyharmonic equations of the fourth order. Mat. Sb. (N.S.) **50 (92)** (1960), 335–368. (Russian)

Consider the Dirichlet problem in a bounded 3-dimensional domain D with a smooth boundary S: $\Delta^2 u = 0$ in D, $u = f_0$ and $\partial u/\partial n = f_1$ on S. Lopatinskiĭ [Ukrain. Mat. Ž. **5** (1953), 123–151; MR **17**, 494], trying to write down the solution as a sum of two potentials with unspecified densities, has reduced the problem to that of solving two Fredholm integral equations with integrable kernels. The present author states that Lopatinskiĭ's integral equations are incomplete since certain terms are missing in one of the integral equations. He rederives the correct equations. He further shows how the equation $\Delta^2 u + a\Delta u + bu = 0$ (a, b constants, $\lambda^2 + a\lambda + b$ having nonnegative roots) can be handled similarly. *A. Friedman* (Minneapolis, Minn.)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also B5117, B5241.

4897:

Bickley, W. G.; McNamee, J. Matrix and other direct methods for the solution of systems of linear difference equations. Philos. Trans. Roy. Soc. London Ser. A **252** (1960), 69–131.

This is a detailed discussion of an attempt to replace by direct methods the approximation methods (such as those of Southwell and Thom) for the solution of the systems of algebraic equations which approximate linear partial differential equations. Matrix methods predominate; many well-chosen worked examples are given. The control of errors is discussed carefully.

The first part is concerned with initial value problems for ordinary differential equations; the second deals with boundary value problems for linear ordinary differential equations. The third, the kernel of the paper, deals with closed methods for the solution of linear partial difference equations in two variables. In considering, for instance, a Dirichlet problem in a square, it has been customary to use the "big matrix" technique. This replaces the continuous problem by a matrix equation $Ax = b$ where x is an n^2-dimensional vector containing the unknown values at the n^2 interior points, where A is a very sparse matrix and b is given by the boundary condition. The authors instead use an equation of the form (1) $AZ + ZB = F$ where Z is an $n \times n$ matrix containing the n^2 unknown values as elements, A and B are matrices corresponding to the x- and y-differential operators, and F is given by the boundary conditions. Various forms of solution to (1) are described and their use in practice is illustrated. There are also accounts of the use of particular solutions generated by factorial polynomials, of a discrete analogue of the kernel function of S. Bergman and of eigenvalue problems.

The fourth part is concerned with the study of initial value problems; it includes a "bad example" warning against implicit reliance in implicit procedures. The method just described for the Dirichlet problem is also applicable here; however the basic equation may be of the form $AZ + CZB = F$. *John Todd* (Pasadena, Calif.)

4898:

Беллман, Р. [Bellman, Richard]. ★Динамическое программирование [Dynamic programming]. Translated from the English by I. M. Andreeva, A. A. Korbut, I. V. Romanovskiĭ, I. N. Sokolova; edited by N. N. Vorob'ev. Izdat. Inostr. Lit., Moscow, 1960. 400 pp. 17.60 r.

For a review of the original edition [Princeton Univ. Press, Princeton, N.J., 1957] see MR **19**, 820.

4899:

Bellman, Richard. Functional equations in the theory of dynamic programming. XI. Limit theorems. Rend. Circ. Mat. Palermo (2) **8** (1959), 343–345.

The author considers the functional equation

$$f_n(p) = \max_Q [b + f_{n-1}(p_1)],$$

where p takes on a finite number of different values, $p \to p_1$ is one of a finite class Q of admissible transformations, and $0 \leqq b$ depends upon p and p_1. He shows that if Q is well-populated enough to transfer any point p into any other in at most $m < +\infty$ steps, then $n^{-1} f_n(p)$ converges as $n \uparrow +\infty$ to a constant independent of p.

H. P. McKean, Jr. (Cambridge, Mass.)

SEQUENCES, SERIES, SUMMABILITY

4900:

Pati, T.; Ahmad, Z. U. On the absolute summability factors of infinite series. II. Indian J. Math. **2**, 29–39 (1960).

Let (ε_n) be a sequence, its kth difference is defined by

$$\Delta^k \varepsilon_n = \sum_{\nu=0}^{\infty} A_\nu^{-k-1} \varepsilon_{\nu+n} \quad (n = 1, 2, \cdots).$$

Let $\sum a_n$ be an infinite series and s_n^k be the nth Cesàro mean of order k; then the authors prove that if $s_n^k = O(\lambda_n)$ where $\lambda_n \downarrow 0$ and $0 < k < 1$, and if (ε_n) is a sequence such that (i) $\sum \lambda_n |\varepsilon_n| < \infty$ and (ii) $\sum n^k \lambda_n |\Delta^{k+1} \varepsilon_n| < \infty$, then the series $\sum a_n \varepsilon_n$ is summable $|C, k|$. This theorem is a generalization of a part of a theorem of L. S. Bosanquet and H. C. Chow [J. London Math. Soc. **32** (1957), 73–82; MR **18**, 733]. Proof is direct and straightforward.

S. Izumi (Budapest)

4901a:

Chillingworth, H. R. On matrix transformations of certain sequence spaces. Nederl. Akad. Wetensch. Proc. Ser. A **60** = Indag. Math. **19** (1957), 578–583.

4901b:

Chillingworth, H. R. "On matrix transformations of certain sequence spaces"—correction and note. Nederl. Akad. Wetensch. Proc. Ser. A **61** = Indag. Math. **20** (1958), 316–318.

The author considers certain matrix transformation spaces and, in particular, the transformations on the space of bounded sequences. He proves that $\sigma_\infty \to \sigma_r$ $(r \geqq 1)$ is the set of matrices (a_{nk}) such that

$$\sum_{n=1}^{\infty} \Big| \sum_{k \in E} a_{nk} \Big|^r \leqq M^r$$

where E is any subset of the positive integers and M is independent of E. Further that $\sigma_\infty \to E_r$ is the set of (a_{nk}) such that $\sum_k |a_{nk}| \leqq N n^r$ for some $N > 0$ and all n. $\sigma_r \to F_r$ is characterized as (a_{nk}) such that $\sum_{n=1}^{\infty} n^r \big| \sum_{k \in E} a_{nk} \big|$ converges for any set E of positive integers. Theorem IV of the paper is not valid for all convergence free spaces but only for certain types, and is replaced in the correction by the following: Let α be a normal sequence space which contains a sequence with no zero terms and let β be convergence free; then (a_{nk}) is in $\alpha \to \beta$ if and only if (i) the row suffixes of the non-zero rows of A form a W-set for β, and (ii) rows of A are in α^*. A corollary is given which replaces theorem V of the first paper.

D. Waterman (Milwaukee, Wis.)

4902:

Li, Syun'-czin'. Cesàro summability in a Banach space. Acta Math. Sinica **10** (1960), 41–54. (Chinese. Russian summary)

Let α be a non-negative real number and E a Banach space. Let $\|u\|$ denote the norm of the element u of E. Let $u_i \in E$ $(i = 0, 1, 2, \cdots)$. Then the nth Cesàro mean of order α of the series $\sum u_i$ is

$$\sigma_n^\alpha(U) = \frac{1}{A_n^\alpha} S_n^\alpha(U),$$

where

$$S_n^\alpha(U) = \sum_{i=0}^{n} A_{n-i}^\alpha u_i, \quad A_n^\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$

If $\lim \sigma_n^\alpha(U) = u$, $u \in E$, then we say that the series (U) is summable (C, α) to u. At the same time, the series $\sum i u_i$ is also studied. Let $\sigma_n^\alpha(T)$ be the nth Cesàro means of order α of this second series. G. Alexits [Acta Math. Acad. Sci. Hungar. **3** (1952), 29–42; MR **14**, 370] has proved that for $\alpha = 1$, the necessary and sufficient condition for the establishment of the relation $\|u - \sigma_n^\alpha(U)\| = O(1/(n+1))$ is that $\|\sigma_n^\alpha(T)\| = O(1)$. J. Favard [J. Math. Pures Appl. (9) **36** (1957), 359–372; MR **19**, 956] has extended this theorem to the case $\alpha = k$, k being a positive integer. In this paper, the author extends Favard's result further to the case α any positive real number.

F. C. Hsiang (Taipeh)

4903:

Rubel, L. A. Maximal means and Tauberian theorems. Pacific J. Math. **10** (1960), 997–1007.

If $0 \leqq \xi < 1$, the Pólya means $L(\xi)$, $l(\xi)$ of a bounded measurable function $f(x)$ are the upper and lower limits as $x \to \infty$ of $[x(1-\xi)]^{-1} \int_{\xi x}^{x} f(t)dt$. In particular, $L(0)$, $l(0)$ are the upper and lower Cesàro means, and it is shown that $L(1) = \lim L(\xi)$, $l(1) = \lim l(\xi)$ exist as $\xi \to 1-$, and that $L(\xi)$, $l(\xi)$ are then continuous in $0 \leqq \xi \leqq 1$. The Littlewood means $\Lambda(p)$, $\lambda(p)$ are defined for $-1 < p < \infty$ as the upper and lower limits as $x \to 0+$ of

$$x^{p+1}[\Gamma(p+1)]^{-1} \int_0^\infty t^p e^{-xt} f(t)\, dt.$$

In particular, $\Lambda(0)$, $\lambda(0)$ are the upper and lower Abel means, and it is shown that $\Lambda(\infty)$, $\lambda(\infty)$, $\Lambda(-1)$, $\lambda(-1)$ exist as limits as $p \to \infty$, $p \to -1+$, respectively, and that $\Lambda(\xi)$, $\lambda(\xi)$ are then continuous in $-1 \leqq p \leqq \infty$.

The main result is that $L(1) = \Lambda(\infty)$, $l(1) = \lambda(\infty)$, and this includes, as an immediate corollary, Littlewood's Tauberian theorem that a bounded measurable function is A-averageable $[\Lambda(0) = \lambda(0)]$ if and only if it is C-averageable $[L(0) = l(0)]$.

It is shown also that the mean $L(1)$ of $f(x)$ is the infimum of the C-averages of all C-averageable functions $f^*(x)$ satisfying $f^*(x) \geqq f(x)$ for all x, and that this infimum is actually attained by a function $f^*(x)$.

H. R. Pitt (Nottingham)

APPROXIMATIONS AND EXPANSIONS

See also 4891.

4904:

Gopengauz, I. E. The deviation of a function from the Lagrange-Hermite interpolating polynomial. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 297–308. (Russian)

The author continues the work of Brudnyĭ and himself [Dokl. Akad. Nauk SSSR **113** (1957), 12–15; MR **19**, 852] and proves the following. For each matrix $A = \{t_k^{(n)}\}$, $k = 0, 1, \cdots, n$, $n = 1, 2, \cdots$, of points in $[a, b]$, there exists a continuous function $f(x)$ such that the interpolation polynomials $Q_n(x)$ with the nodes $t_k^{(n)}$, $k = 0, 1, \cdots, n$, have the following singularity: the set of points

of $[a, b]$, where $|f(x)-Q_n(x)|$ attains its maximum has positive measure for each n. For some A (for example, for the matrix of the zeros of the Chebyshev polynomials) the function $f(x)$ can be assumed to be infinitely differentiable. *G. G. Lorentz* (Syracuse, N.Y.)

4905a:

Moldovan, Elena. Propriétés des fonctions convexes généralisées. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **8** (1957), 21–35. (Romanian. Russian and French summaries)

4905b:

Moldovan, Elena. Fonctions à variation bornée. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **8** (1957), 313–317. (Romanian. Russian and French summaries)

4905c:

Moldovan, Elena. Sur une notion de fonction convexe par rapport à un ensemble de fonctions interpolatrices. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **9** (1958), 161–224. (Romanian. Russian and French summaries)

In these three papers the author considers families F_n of continuous, real-valued functions on a real interval $[a, b]$; such a family F_n is called n-interpolatory whenever for each sequence $x_1, x_2, \cdots, x_n$ of distinct points of $[a, b]$ and each function f on $[a, b]$ there is a unique φ in F_n such that $\varphi(x_i)=f(x_i)$ $(i=1, 2, \cdots, n)$. A function f on $[a, b]$ is called convex, non-concave, polynomial, non-convex, or concave relative to F_n if $f(x_{n+1})-\varphi(x_{n+1})>$, $\geqq$, $=$, $\leqq$, or <0 for each sequence $x_1<x_2<\cdots<x_n<x_{n+1}$ of points of $[a, b]$.

The first paper discusses properties of n-interpolatory families F_n and of functions convex relative to them. An example is theorem 11: If $F_1 \subset F_2 \subset \cdots \subset F_n$ and if for each k with $1 \leqq k \leqq n-1$, $n \geqq 2$ the family F_k is k-interpolatory on $[a, b]$, then for each f convex relative to F_n on $[a, b]$ and each $k<n$ there is a partition of $[a, b]$ into $k+1$ non-overlapping intervals in which f is alternatively convex and concave relative to F_{n-k}.

The second paper defines the total variation of f relative to F_n and gives some results analogous to classical properties of functions of bounded variation.

Chapter 1 of the last paper gives further properties including "mean-value" theorems of functions of F_n. Chapter 2 discusses properties related to that quoted from the first paper. Chapter 3 gives differential inequalities characterizing the functions related to F_n. Chapter 4 studies Čebyšev approximation relative to F_n.

M. M. Day (Urbana, Ill.)

4906:

Edwards, Robert E. An approximation problem proposed by K. Itô. Tôhoku Math. J. (2) **11** (1959), 406–408.

For certain sequences $\{C_n\}$ and exponents p, Izumi [same J. **5** (1953), 22–28; MR **15**, 217] has proved Itô's conjecture that for any $\varepsilon>0$ and any function f obeying $\sum C_n|f(n)|^p<\infty$, a polynomial P can be found such that $\sum C_n|f(n)-P(n)|^p<\varepsilon$. In the present paper, the author obtains a much more general theorem using the non-constructive methods of functional analysis. Let T be a locally compact space, homeomorphic with E^n under a mapping $t\to(u_1(t), \cdots, u_n(t))$. Assume also that μ is a positive Radon measure on T such that

$$\int_T \exp(C\sum_1^n |u_j(t)|)\, d\mu(t) < \infty$$

for some $C>0$. Then, he proves that the functions $p(t)=P(u_1(t), u_2(t), \cdots, u_n(t))$, where P ranges over all polynomials, are dense in $L^p(T, \mu)$ for any $1 \leqq p<\infty$. The method is the familiar one of looking for linear functionals on $L^p(T, \mu)$ that annihilate all the $p(t)$.

R. C. Buck (Madison, Wis.)

4907:

de Branges, Louis. The Bernstein problem. Proc. Amer. Math. Soc. **10** (1959), 825–832.

If $w(x)$ is a positive continuous weight function which vanishes strongly at $\pm\infty$, the Bernstein problem asks if the linear span of $\{x^n w(x)\}_{n=0}^{\infty}$ is uniformly dense in $C_0[-\infty, \infty]$. A satisfactory complete solution was found by Fuchs and Pollard [same Proc. **6** (1955), 613–615; MR **17**, 255]. The present paper presents another necessary and sufficient condition, based upon the non-existence of an entire function F of exponential type, with real simple zeros λ_n, and not a polynomial, such that

$$\int (1+t^2)^{-1} \log^+ |F(t)| < \infty$$

and $\sum 1/|F'(\lambda_n)|w(\lambda_n)<\infty$. *R. C. Buck* (Madison, Wis.)

4908:

Stancu, D. D. De l'approximation, par des polynômes du type Bernstein, des fonctions de deux variables. Com. Acad. R. P. Romîne **9** (1959), 773–777. (Romanian. Russian and French summaries)

The author extends known results for functions of one variable to functions of two variables. The two-dimensional Bernstein polynomials B_n converge uniformly to a continuous f in $x+y\leq 1$, $x\geq 0$, $y\geq 0$; the difference $|f-B_n|$ is at most $2\omega(n^{-1/2})$; if f has continuous second derivatives then

$$n[B_n-f] \to \tfrac{1}{2}x(1-x)f_{xx}-xyf_{xy}+\tfrac{1}{2}y(1-y)f_{yy};$$

if $\partial^{r+s}f(x, y)/\partial x^r \partial y^s$ is continuous, with modulus of continuity $\omega_{r,s}$ and maximum absolute value $M_{r,s}$, then it is approximated by the same derivative of B_n within

$$\{1+[1+2(r+s)]^{1/2}\}\omega_{r,s}((n-r-s)^{-1/2}) + \tfrac{1}{2}n^{-1}(r+s)(r+s-1)M_{r,s}.$$

R. P. Boas, Jr. (Evanston, Ill.)

4909:

Roşcău, H. Sur certains problèmes d'osculation et quelques types d'équations différentielles. Inst. Politehn. Cluj. Lucrări Şti. **1959**, 83–93. (Romanian. Russian and French summaries)

If two functions $f(z)$ and $g(z)$ are regular at $z=x$ and such that their difference $h(z)=f(z)-g(z)$ together with its first n derivatives vanishes for $z=x$, $h^{(i)}(x)=0$ $(i=0, 1, \cdots, n)$, while $h^{(n+1)}(x)\neq 0$, then the functions $f(z)$ and $g(z)$ are said to be osculating of order n at x. The author treats the problems: (a) of determining a rational function of the form

$$g_1(z) = A_0+\sum_{i=1}^{m} A_i(1-(z-x)/\alpha)^{-i}+\sum_{i=1}^{p} B_i(1-(z-x)/\beta)^{-i}$$

or of the form

$$g_2(z) = A_0 + \sum_{i=1}^{n} A_i(1-(z-x)/\alpha_i)^{-1},$$

osculating to a given function $f(z)=\sum_{n=0}^{\infty} f^{(n)}(x)(z-x)^n/n!$, regular at x; and (b) of determining the functions $f(x)$ so that the rational functions $g(z)$, osculating of order n to $f(z)$ at $z=x$, should have their poles independent of x. The results are as follows: there are at most $2mp^2$ functions $g_1(z)$ osculating of order $n=m+p+2$ to $f(z)$ at $z=0$ {the dissymmetry between p and m is not clear to the reviewer}; (2) the most general function $f(z)$, such that $g_1(z)$, with fixed poles α_1, β_1, of orders m and p, respectively, should be osculating to it of order $m+p+2$ at x, is also rational, with the same poles, of same order as $g_1(z)$, and depending on a certain number of arbitrary constants; it satisfies a certain (complicated) differential equation. Similar results hold also for the form $g_2(z)$.

E. Grosswald (Philadelphia, Pa.)

4910:

Roşcău, H. L'approximation des fonctions à l'aide des fractions rationnelles. Inst. Politehn. Cluj. Lucrări Şti. **1959**, 95–100. (Romanian. Russian and French summaries)

If the values of a function $f(x)$ are given at the $m+n$ points x_i $(i=0, 1, \cdots, m+n-1)$, one can (in general) determine a rational approximation

$$u(x) = \Big(\sum_{k=0}^{n-1} a_k x^k\Big) \Big/ \Big(\sum_{j=0}^{m} b_j x^j\Big)$$

of $f(x)$, where the a_k's and b_j's are uniquely determined (except for an arbitrary factor). The explicit formula for the a_k's and b_j's as functions of the given abscissae and functional values is due to Cauchy. The author indicates a proof of Cauchy's theorem and also (under the assumption that $f(x)-u(x)$ is continuously differentiable $m+n-1$ times and that its $(m+n)$th derivative exists and is finite) finds an expression for the error term, by showing that $f(x)-u(x)=k\prod_{i=0}^{m+n-1}(x-x_i)$, with

$$k = (1/(m+n)!)\,\{f^{(m+n)}(\xi)-u^{(m+n)}(\xi)\}$$

and $m \leqq \xi \leqq M$, where m and M stand for the least and largest, respectively, among the values $x, x_0, x_1, \cdots, x_{m+n-1}$. There is no mention of reciprocal differences.

E. Grosswald (Philadelphia, Pa.)

4911:

Coţiu, A. Sur une formule de quadrature de deux nœuds doubles. Inst. Politehn. Cluj. Lucrări Şti. **1959**, 41–48. (Romanian. Russian and French summaries)

This is a study of the quadrature formula

$$\int_a^b f(x)dx = h[(1-3u^2)f(a)+2f(x_1)+2f(x_2) + (1-3u^2)f(b)]/3(1-u^2)+\int_a^b \phi(x)f^{iv}(x)dx,$$

$x_1=\frac{1}{2}(a+b)-hu$, $x_2=\frac{1}{2}(a+b)+hu$, $h=\frac{1}{2}(b+a)$, in which the coefficients, nodes, and remainder depend on the parameter u (a value in the interval $(0, 1)$), for the limit value $u=1$. In particular, if $u=0$, this formula reduces to Simpson's rule. If $u=\frac{1}{3}$, Newton's formula is obtained. The remainder is also studied from a limit superior for the absolute value of the remainder. A comparison is made between this quadrature formula and others of the same degree of exactness.

E. Frank (Chicago, Ill.)

4912:

Coţiu, A. Quelques formules pour l'intégration numérique des équations différentielles du premier ordre déduites à l'aide des différences. Inst. Politehn. Cluj. Lucrări Şti. **1959**, 49–59. (Romanian. Russian and French summaries)

Formulas are obtained for the numerical integration of the first order differential equation $y'=f(x, y)$ with the initial condition $y(x_0)=y_0$. This is done by the replacement of the derivatives by difference approximations in the series development $y(x_{i+1})=y(x_i+h)=y(x_i)+hy'(x_i)/1!+h^2y''(x_i)/2!+\cdots$. Then the integration formulas are $y_{i+1}=y_i+h(k_1+4k_2+k_3)/2$, where $k_1=f(x_i, y_i)$, $k_2=f(x_{i+1/2}, y_i+hk_1/2)$, $k_3=f(x_{i+1}, y_i+hk_1)$; and also

$$y_{i+1} = y_i+h\{f(x_i, y_i)+3f(x_i+h/3, y_i+hf(x_i, y_i)/3) + 3f(x_i+2h/3, y_i+2hf(x_i, y_i)/3) + f(x_i+h, y_i+hf(x_i, y_i))\}/8.$$

Estimates of the errors of these formulas are given and compared, and the formulas are generalized.

E. Frank (Chicago, Ill.)

4913:

Korobov, N. M. Computation of multiple integrals by the method of optimal coefficients. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1959**, no. 4, 19–25. (Russian)

The author uses the following notation: $s \geqq 1$, $N > s$ and $a_i=a_i(N)$ are integers; $\alpha \geqq 1$, $C=C(\alpha, s)>0$ and $\beta=\beta(\alpha, s)$ are real numbers; $\{x\}$ is the fractional part of the number x. For all functions $f(x_1, \cdots, x_s)$ belonging to a certain class $E_s(\alpha)$, let the following estimate

$$\left|\int_0^1 \cdots \int_0^1 f(x_1, \cdots, x_s)\,dx_1 \cdots dx_s - \frac{1}{N}\sum_{k=1}^{N} f\left(\left\{\frac{ka_1}{N}\right\}, \cdots, \left\{\frac{ka_s}{N}\right\}\right)\right| \leqq CN^{-\alpha}\ln^{\beta} N$$

be fulfilled for an infinite sequence of values of N.

We say that the above estimate 'does not admit an essential improvement' if for no choice of $a_1, \cdots, a_s$ is it permitted—for all functions of the class $E_s(\alpha)$ and the same values of N—to replace α by α', where $\alpha' > \alpha$. The numbers $a_1, \cdots, a_s$ corresponding to an estimate which does not admit an essential improvement are called 'optimal' coefficients. In a previous work [Dokl. Akad. Nauk SSSR **124** (1959), 1207–1210; MR **21** #2848] the author proposed for a certain class a recursion process for finding numbers $a_1, \cdots, a_s$ such that if N is prime the above estimate has the form $O(N^{-\alpha}\ln^{\alpha} N)$ and does not admit an essential improvement.

In the present paper a method is proposed for finding optimal coefficients which yield an estimate of the form $O(N^{-\alpha})$ for the same class. He then considers an extension of that class and derives for it an estimate also of the form $O(N^{-\alpha})$. He points out that the method of uniform nets when applied to this class of functions yields only an estimate of the form $O(N^{-\alpha/s})$.

A. H. Stroud (Madison, Wis.)

4914:

Chen, Kien-kwong. Summability of the series of orthogonal polynomials. Acta Math. Sinica **10** (1960), 33–40. (Chinese. English summary)

Let $\tau(x)$ be a positive integrable function defined in the interval (a, b) which determines a sequence of functions $\{P_n(x)\}$ $(n=0, 1, 2, \cdots)$ such that

$$\int_a^b P_n(x)P_m(x)\tau(x)\,dx = 1 \quad (n=m),$$
$$= 0 \quad (n \neq m).$$

Let $f(x)$ be expanded into the series

$$(1) \quad f(x) \sim \sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \int_a^b f(x)P_n(x)\tau(x)\,dx.$$

In this note, the known Natanson theorem [Izv. Akad. Nauk SSSR. Ser. Fiz.-Mat. **1** (1933), 85–88] that if $f(x) \in \text{Lip } \alpha$, $\alpha > \frac{1}{2}$, then (1) is almost everywhere convergent, is extended as follows. Theorem 1: If the modulus of continuity of $f(x)$, $\omega(f, t)$ satisfies

$$\int_0^{1/e} \omega^2(f, t) \log \frac{1}{t} \frac{dt}{t} < \infty,$$

then (1) is almost everywhere convergent in (a, b). Theorem 2: If

$$\int_0^{1/e} \omega^2(f, t) \frac{(\log^2 t^{-1})^2}{\log t^{-1}} \cdot \frac{dt}{t} < \infty,$$

then $f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$ (C, α) almost everywhere in (a, b).

F. C. Hsiang (Taipeh)

4915:

Gupta, D. P. An aspect of local property for convergence of ultraspherical series. Indian J. Math. **1**, 55–60 (1959).

It is shown that the convergence behaviour of the Fourier series $\sum A_n P_n^{(\lambda)}(x)$ of a function $f(x) \in L(-1, 1)$ (in terms of Gegenbauer polynomials $P_n^{(\lambda)}(x)$ of order λ) at any point x $(-1 < x < 1)$ depends, under the assumption $A_n = o(n^{1-\lambda})$, only on the behaviour of $f(x)$ in the "immediate neighborhood" of x, an extension of W. H. Young's theorem for $\lambda = \frac{1}{2}$.

K. Endl (Columbus, Ohio)

FOURIER ANALYSIS

See also B5241, B5242.

4916:

Yano, Kenji. Notes on uniform convergence of trigonometrical series. II. Proc. Japan Acad. **35** (1959), 257–262.

The author continues part I [same Proc. **35** (1959), 197–200; MR **21** #6501]. Let $A_n^\gamma = \binom{\gamma+n}{n}$,

$$t_n^\gamma = \sum_0^n A_{n-\nu}^{\gamma-1}(\nu a_\nu) = \sum A_{n-\nu}^{\gamma-1} t_\nu$$

$(-\infty < \gamma < \infty)$, a_ν reals. He proves three theorems. The following is representative of the results proved. Let $0 < s \leq 1$ and q be an arbitrary real constant. If

$$(1-x)\sum_1^{\infty} n a_n x^n \to 0 \quad (x \to 1-0),$$

$$\sum_n^{2n} (|\gamma_n| - \gamma_n) = O(n^{1-s}) \quad (n \to \infty),$$

where $\gamma_n = (1+qn^{-1}) \cdot t_n^{1-s} - t_{n+1}^{1-s}$ $(n=1, 2, \cdots)$, then (I) $\sum a_n \sin nt$ converges uniformly and (II) $\sum a_n \cos nt$ converges if and only if $\sum a_n$ converges.

M. Tomić (Belgrade)

4917:

Še, Din-fan. Absolutely convergent Fourier series. Sci. Record (N.S.) **2** (1958), 439–442. (Russian)

Define

$$\Delta_h^{(l)} f(x) = \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} f(x+jh).$$

Theorem 1: Let $1 < p \leq 2$, $0 < \beta < p/(p-1)$, $\gamma \geq 0$, $N > 1$. Suppose

$$\sum_{n=1}^{\infty} n^{\gamma+1-\beta} \{N^n \int_0^{2\pi} |\Delta_{2\pi/N^n}^{(l)} f(x)|^p dx\}^{\beta/p}$$

converges. Then $\sum_{n=1}^{\infty} (|a_n|^\beta + |b_n|^\beta) n^\gamma$ converges, a_n and b_n the Fourier coefficients of the 2π-periodic function f. Theorem 2: Let $0 < \beta < 2$, $\gamma \geq 0$, and let f be continuous. Suppose

$$\sum_{n=1}^{\infty} 2^{n(\gamma+1-\beta)} \{ \max_{|h| \leq 2^{-n}} \max_x \Delta_h^{(l)} f(x)\}^{\beta/2}$$

converges. Then same conclusion as in theorem 1. Theorems 3 and 4 are of a similar nature, for double Fourier series. The paper contains no proofs.

H. Mirkil (Hanover, N.H.)

4918:

Geronimus, Ya. L. On the order of approximation by means of Poisson's integral. Dokl. Akad. Nauk **129** (1959), 726–728. (Russian)

Let $f(\theta)$ be a periodic integrable function and let $F(re^{i\phi})$ be its Poisson integral; $\Delta(r, \phi) = F(re^{i\phi}) - f(\phi)$; $w_\phi(t) = f(\phi+t) + f(\phi-t) - 2f(\phi)$; $W_\phi(\delta) = \sup_{|t| \leq \delta} |w_\phi(t)|$. Suppose that f is continuous at ϕ or has a simple jump there. The author proves: If $\int_0^\pi |w_\phi(t)| t^{1-\gamma} dt$ converges, where $0 < \gamma \leq 1$, then for r sufficiently near 1 we have (with various constants C)

$$|\Delta(r, \phi)| \leq C(1-r)^\gamma;$$
$$|\Delta(r, \phi)| \leq C\delta^{-2}(1-r), \qquad 1-r^2 = \delta^2 W_\phi(\delta);$$

and if $\int_0^\delta |w_\phi(t)| dt \leq \delta^2 \mu(\delta)$, $\mu(\delta) \downarrow 0$, then

$$|\Delta(r, \phi)| \leq C\delta^{-2}(1-r)^2, \qquad 1-r = \delta^2 \mu(\delta)^{1/2}.$$

Now let E be a set of positive measure on which $f'(\theta)$ exists (finite). Then for $\phi \in E$ and r sufficiently near 1,

$$|\Delta(r, \phi)| \leq C(1-r).$$

Let f be continuous on $[\alpha, \beta]$ with modulus of continuity $\omega(\delta)$. Then if ϕ, ψ are in an interior interval and r is near 1,

$$|\Delta(r, \phi)| \leq \omega(|\phi - \psi|) + C\omega\left\{(1-r)\log\frac{1}{1-r}\right\}.$$

Application: If $P(z)$ is of class H_1 and its boundary function has modulus of continuity $\omega(\delta)$ on $[e^{i\alpha}, e^{i\beta}]$, then

with $z=e^{i\phi}$, $z_1=re^{i\psi}$, with ϕ and ψ in an interior interval,

$$|P(z)-P(z_1)| \leq C\omega\left\{|z-z_1|\log\frac{1}{|z-z_1|}\right\}.$$

R. P. Boas, Jr. (Evanston, Ill.)

4919:

de Branges, Louis. The a-local operator problem. Canad. J. Math. **11** (1959), 583–592.

Let M be the space of functions f that are Fourier transforms $f(t)=\int e^{itx}\,d\mu(x)$ of finite Borel measures μ. For any Borel measurable function K, define an operator $K(H)$ by $K(H)\{f\}(t)=\int e^{itx}K(x)\,d\mu(x)$; the domain of $K(H)$ can be taken as all $f\in M$ with $\int|K(x)\,d\mu(x)|<\infty$. For any $a>0$, one says that an operator T is a-local if $T(f)(0)=0$ for every f in the domain of T which vanishes on $|t|\leq a$. The author obtains a partial solution to the problem of determining when an operator $K(H)$ is a-local. If $K(H)$ is a-local, then $K(x)$ can be extended to an entire function which obeys $h(\theta, K)\leq a|\sin\theta|$ and

$$\int(1+t^2)^{-1}\log^+|K(A)|\,dt<\infty.$$

Conversely, if in addition one assumes that $K(iy)=O(e^{a|y|})$ for all z, then $K(H)$ is a-local. Several alternative conditions are also discussed.

R. C. Buck (Madison, Wis.)

4920:

Koosis, Paul. Sur la totalité des systèmes d'exponentielles imaginaires. C. R. Acad. Sci. Paris **250** (1960), 2102–2103.

On sait qu'il existe des suites positives $\{\lambda_k\}$, de densité nulle, telles que le système $\{e^{i\lambda_k x}\}$ soit total sur tout intervalle de la droite [le référent, Ann. Inst. Fourier Grenoble **8** (1958), 273–275; MR **21** #5115]; mais les suites construites par le référent sont loin de satisfaire la condition $\inf(\lambda_{k+1}-\lambda_k)>0$. L'auteur construit une suite d'entiers $\{\lambda_k\}$ de densité 0 telle que le système $\{e^{i\lambda_k x}\}$ n'est total sur aucun intervalle de longueur $<2\pi$; la construction repose sur une nouvelle condition de totalité.

J.-P. Kahane (Montpellier)

4921:

Bredihina, E. A. Some estimates of the deviation of partial sums of Fourier series of almost-periodic functions. Mat. Sb. (N.S.) **50 (92)** (1960), 369–382. (Russian)

Let (1) $f(x)=\sum A_k\exp(i\lambda_k x)$ be an almost periodic function whose sequence of exponents $\Lambda=\{\lambda_k\}$ has no finite point of accumulation; $\lambda_0=0$, $-\lambda_{-k}=\lambda_k<\lambda_{k+1}$ $(k=1, 2, \cdots)$. Put

$$R(t,f)=\sup_x|f(x)-\sum_{|\lambda_k|\leq t}A_k\exp(i\lambda_k x)|,$$

$E(t,f)=$ best Chebychev approximation of f by entire functions of type $\leq t$ bounded on the real axis. Let $n(x)$ be the number of λ_k satisfying $0<\lambda_k\leq x$. Theorem 1: If $0<t<\mu$, then

$$R(t,f)\leq E(t,f)\left[1+\frac{4}{\pi}+2\{n(\mu)-n(t)\}+\frac{2}{\pi}\log\frac{\mu+t}{\mu-t}\right].$$

Several results of the form $R(t,f)\leq\phi(t,\Lambda)E(t,f)$ are deduced from theorem 1 for special types of sequences Λ.

Putting $\mu=\lambda_{n+1}$, $t=\lambda_n$ and observing that $E(t,f)\to0$ as $t\to\infty$, the author obtains theorem 2: The Fourier expansion of $f(x)$ is uniformly convergent, if

$$\lim_{n\to\infty}E(\lambda_n,f)\cdot\log\frac{\lambda_{n+1}+\lambda_n}{\lambda_{n+1}-\lambda_n}=0.$$

Several other tests of uniform convergence based on theorem 1 are given.

Generalizing her earlier results [Dokl. Akad. Nauk SSSR **111** (1956), 1163–1166; MR **18**, 886], the author proves theorem 3: If, for some integer s, $\lambda_{k+s}/\lambda_k>\theta>1$ $(k=1, 2, \cdots)$, then the series (1) is absolutely convergent, and if $A_0=0$, then $\sum|A_k|<C(\Lambda)\sup_x|f(x)|$.

W. H. J. Fuchs (Ithaca, N.Y.)

4922:

Rudin, Walter. Closed ideals in group algebras. Bull. Amer. Math. Soc. **66** (1960), 81–83.

Utilisant une idée de Malliavin [C. R. Acad. Sci. Paris **248** (1959), 1756–1759, 2155–2157; MR **21** #5854a–b], l'auteur prouve le théorème suivant: soit G un groupe abélien compact infini, et $A(G)$ l'algèbre de Banach des fonctions définies sur G et transformées de Fourier de fonctions sommables sur le groupe dual; il existe une $f\in A(G)$, réelle, telle que les idéaux fermés engendrés par les f^n $(n=1, 2, \cdots)$ sont tous distincts.

J.-P. Kahane (Montpellier)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

4923:

Srivastava, Krishna Ji. Self-reciprocal function and $\varpi_{\mu,\nu}$-transform. Bull. Calcutta Math. Soc. **51** (1959), 57–65.

The set $R_{\mu,\nu}$ of self-reciprocal functions in $L^2(0,\infty)$ is investigated for a certain transformation $g=Tf$ {which, the reviewer remarks, by known general results is the Faltung $T_\mu QT_\nu$ of two Hankel transforms T of orders μ and ν ($\operatorname{Re}\mu>-\frac{1}{2}$, $\operatorname{Re}\nu>-\frac{1}{2}$), with $Q\{F(t)\}=t^{-1}F(t^{-1})$}. Let $0<z<\infty$, $f(t)\in L^2(0,\infty)$,

$$I^+_{\eta,\alpha}f=\int_0^z\frac{(z-t)^{\alpha-1}t^\eta}{\Gamma(\alpha)z^{\eta+\alpha}}f(t)\,dt,$$

$$K^-_{\eta,\alpha}f=\int_z^\infty\frac{(t-z)^{\alpha-1}z^\eta}{\Gamma(\alpha)t^{\eta+\alpha}}f(t)\,dt,$$

$\operatorname{Re}\eta>-\frac{1}{2}$, $\operatorname{Re}\alpha>0$. It is shown that the operator

$$\{I^+_{\eta,\alpha}\}^m\{K^-_{\eta,\alpha}\}^n+\{I^+_{\eta,\alpha}\}^n\{K^-_{\eta,\alpha}\}^m \qquad (m=0, 1, 2, \cdots;\ n=0, 1, 2, \cdots)$$

maps $R_{\mu,\nu}$ to a subspace of $R_{\mu,\nu}$. Then a transformation $\mathfrak{S}_{\mu,\nu}$ is discussed {which, in fact, is merely the operator $T_\mu{}^*QT_\nu{}^*$, where T^* is the Hankel operator in Tricomi's form}, and finally a mapping due to A. Erdélyi [J. London Math. Soc. **16** (1941) 113–117; MR **3**, 39; p. 116], representable in the form TQ, where T is of the Hankel type, with kernel $K(x)=x^{-\alpha/2}J_{2\eta+\alpha}(2\sqrt{x})$.

{It is, however, not difficult to show that the first result holds when T_μ and T_ν are replaced by any operators of the "General Transforms" type.}

H. Kober (Birmingham)

4924:

Rooney, P. G. On some properties of functions analytic in a half-plane. Canad. J. Math. **11** (1959), 432–439.

Let ω be real, $1<p\leqq 2$, $e^{-\omega t}\phi(t)\in L_p(0,\infty)$. If $f(s)$ is the Laplace transform of $\phi(t)$, Re $s>\omega$, then

$$v_p(f;x,\omega)=(2\pi)^{-1}\int_{-\infty}^{\infty}|x-\omega+iy|^{p-2}|f(x+iy)|^p\,dy$$

is bounded for $x>\omega$; more precisely,

$$v_p(f;x,\omega)\leqq K\int_0^{\infty}e^{-px t}|\phi(t)|^p\,dt,$$

where K depends only on p.—The following further assertion holds. If $f(s)$ is such that $v_q(f;x,\omega)$ is bounded for $x>\omega$, $q\geqq 2$, then there exists a ϕ, with $e^{-\omega t}\phi(t)\in L_q(0,\infty)$, such that f is the Laplace transform of ϕ, Re $s>\omega$. Further, with $x=\omega$,

$$\int_0^{\infty}e^{-qxt}|\phi(t)|^q\,dt\leqq Kv_q(f;x,\omega),$$

where K depends on q alone. Two generalizations are given.

G. Szegö (Stanford, Calif.)

4925:

Spain, B. Interpolated derivatives. Proc. Edinburgh Math. Soc. (2) **9** (1958), 166–167.

The author has previously given a formula for $D^nf(x)$ valid for $R(n)<0$ and has also given the analytic continuation for $R(n)\geqq 0$. He considered the result unsatisfactory in that the repeated application of the operation of fractional differentiation was not possible. Here instead of the cardinal function interpolation, the Gregory-Newton formula is employed and the usual generalization of the fractional derivative,

$$D^{-n}f(x)=\frac{1}{(n)}\int_a^x(x-u)^{n-1}f(u)\,du,$$

is obtained.

D. Waterman (Lafayette, Ind.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

4926:

Rizzonelli, Pieranita. Sulla risoluzione delle equazioni integrali concernenti composizioni secondo varietà lineari. Boll. Un. Mat. Ital. (3) **14** (1959), 327–337. (English summary)

The results of this paper concern themselves with the extension of Dini's theorem of implicit functions to integral equations of the Faltung type, in which compositions of the second kind (Fredholm) are considered. To define this composition more explicitly, consider two functions $F(t)$ and $G(\eta)$ defined respectively in the space τ of N dimensions and in the space σ of K dimensions, where $1\leqq K<N$; let t and η be column vectors of order N and K respectively, where the vector will be called positive or negative according as its components are all positive or all negative; and let $F(t)$ and $G(\eta)$ vanish for $t<0$ and $\eta<0$ respectively. Furthermore, let A be a real matrix of dimension $N\times K$ with positive or zero elements and without a null column, viz., $A=(a_{ij})$ $(i=1,\cdots,N;$ $j=1,\cdots,K)$. Consider next a point t in τ' and the linear manifold V passing through t and having the parametric equations $\xi_i=t_i-\sum_{s=1}^{K}a_{is}\eta_s$ $(i=1,\cdots,N)$. In matrix form the manifold V may be denoted by $\xi=t-A\eta$, where ξ is a column vector in the space τ. Then the product of compositions of the second kind over the manifold V is defined by $G_V*F(t)=\int_\sigma G(\eta)F(t-A\eta)\,d\sigma$, with $G_V*F(t)=0$ for $t<0$.

I. A. Barnett (Cincinnati, Ohio)

4927:

Berg, Lothar. Lösungsverfahren für singuläre Integralgleichungen. II. Wiss. Z. Univ. Rostock. Math.-Nat. Reihe **4** (1954/55), 381–391.

[For part I see Berg, Math. Nachr. **14** (1955), 193–212; MR **19**, 66.]

The paper refers to singular integral equations pertaining to a set of closed, smooth and double-point-free curves in the complex plane. The author first gives a review of known results, as they can be found in Muskhelishvili's book *Singular integral equations* [Noordhoff, Groningen, 1953; MR **15**, 434]. He then quotes some results of his own dissertation (the first part of the paper), which deal with the existence of solving operators.—The author's main goal is to transform the integral equation into an infinite set of ordinary linear equations for the coefficients which appear in certain functional expansions of the solution or solutions. In particular he deals with power expansions and with those of the Enskog method. As for power expansions, it is assumed that in the integral equation $Ku=f$ with the singular operator K and the characteristic operator K^0 the difference $K-K^0$ as well as f admits power expansions in all of the variables concerned, provided that the integral equation has Vecoua's canonical form. The set of equations for the coefficients is considered for the singular equation as well as for regularized forms of it (Carleman-Vecoua form, the author's form of equivalent Fredholm equations), and comparisons are made in which the index of K plays an important role. For the application of the Enskog method the index is assumed to be nonpositive; from a complete orthonormal system of functions $\{u_n\}$ a generalized orthonormal system $\{v_n\}$ to satisfy $(Kv_n, Kv_m)=\delta_{nm}$ is found. Finite linear combinations of the v_n are used to approximate the solution in the norm that accompanies the generalized scalar product. In this connection the author also considers unitary and half-unitary operators and finds results about characteristic unitary operators.—Since a proper formulation of the author's results is rather space-consuming, the reader is referred to the paper for details.

H. Bückner (Madison, Wis.)

4928:

Stewart, Charles E. On the numerical evaluation of singular integrals of Cauchy type. J. Soc. Indust. Appl. Math. **8** (1960), 342–353.

The author is concerned with the numerical evaluation of Cauchy type integrals such as those arising in airfoil theory. Several theorems on the convergence of sequences of Cauchy integrals are proved, the following being representative.

Theorem: Let $\{f_n\}$ be a sequence of functions continuous on $a\leqq x\leqq b$. Let $((a,b))=\{(x,y);\ a<x<b,\ a<y<b,\ x\neq y\}$, and suppose the sequence $F_n(x,y)=(f_n(x)-f_n(y))/(x-y)$ is uniformly convergent on $((a,b))$. The

sequence $\{f_n\}$ is then called secant convergent. Then if for each n the Cauchy principal value

$$f_n^{(1)}(y) = \int_a^b \frac{f_n(x)dx}{(x-y)[(b-x)(x-a)]^{1/2}}$$

exists, the sequence $\{f_n^{(1)}(y)\}$ converges uniformly on (a, b).

Application is made to polygonal approximations for integrals of the form

$$\int_a^b \frac{f(x)dx}{(x-y)[(b-x)(x-a)]^{1/2}} \quad (a < y < b).$$

R. C. MacCamy (Pittsburgh, Pa.)

4929:

Struble, Raimond A. Differentiation of singular integrals of Cauchy type. J. Soc. Indust. Appl. Math. **8** (1960), 305–308.

Let $F(y)=\int_a^b f(x, y)/(x-y)\,dx$. The author establishes conditions for the validity of the formula,

$$\frac{dF}{dy} = \frac{f(a, y)}{a-y} - \frac{f(b, y)}{b-y} + \int_a^b \frac{\partial f(x, y)}{\partial x}\frac{dx}{x-y} + \int_a^b \frac{\partial f(x, y)}{dy}\frac{dx}{x-y},$$

using the notions of secant convergence. [See preceding review.] *R. C. MacCamy* (Pittsburgh, Pa.)

4930:

Vaĭnberg, M. M.; Kacurovskiĭ, R. I. On the variational theory of non-linear operators and equations. Dokl. Akad. Nauk SSSR **129** (1959), 1199–1202. (Russian)

This paper continues the study by Vaĭnberg and his school of the Hammerstein operator $\Gamma = Ah$, where

$$Av = \int_0^1 K(x, y)v(y)\,dy$$

with symmetric kernel K and $hu = g(u(x), x)$ is the so-called Nemyckiĭ operator [see M. M. Vaĭnberg, *Variacionnye metody issledovaniya nelineĭnyh operatorov*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956; MR **19**, 567]. Assume that for some sufficiently large n, the operator $A_{i\nu}$ is self-adjoint and positive in L^2 for $i, \nu = 1, \cdots, n$, where

$$A_{i\nu}v = \int_0^{1/n} K(x+(i-1)/n,\ y+(\nu-1)/n)v(y)\,dy,$$

and also that A is completely continuous from L^q into L^p, $p \geq 2$. Then under suitable growth conditions on g the equation $u = \Gamma u$ has a solution in L^p, and also for any $\varepsilon > 0$ there is a continuum of eigenfunctions of Γ of L^p-norm less than ε. *W. H. Fleming* (Providence, R.I.)

4931:

Nohel, John A. A class of nonlinear delay differential equations. J. Math. and Phys. **38** (1959/60), 295–311.

The problem of finding a function $u(t)$ such that, for $t > \theta$,

$$u' = a\int_0^\theta (\theta-h)u(t-h)\,dh + \int_0^\theta (\theta-h)g(t-h,\ u(t-h))\,dh,$$

and $u(t) = q(t)$ $(0 < t \leq \theta)$, $q(t)$ given, is a generalization of a problem of importance in reactor theory. In this paper the stability, as $t \to \infty$, of the solution $u \equiv 0$ for $q \equiv 0$ is studied under the assumption that $g(t, u)$ is continuous and $g(t, u) = o(|u|)$ as $u \to 0$, uniformly in t, $t \geq 0$. The constant θ is positive, and a is a real parameter. Several theorems are proved, the most important of which states that $u \equiv 0$ is an asymptotically stable solution if $a < 0$ and $a\theta^3 + (2n\pi)^2 \neq 0$ for $n = 0, 1, 2, \cdots$. If $a > 0$, the solution $u \equiv 0$ is unstable. Also included is a slightly sharper result proved under somewhat different assumptions on $g(t, u)$. The proofs resemble in structure those of analogous theorems for differential equations. However, considerable modifications are necessary. A major difficulty is the construction of a formula to take the place of the variation of parameter method in the theory of linear differential equations. This problem the author solves with the help of the theory of the Laplace transformation.

W. Wasow (Madison, Wis.)

FUNCTIONAL ANALYSIS

See also 4651, 4825, 4833, 4883, 4989, B5069.

4932:

Levin, V. L. Non-degenerate spectra of locally convex spaces. Dokl. Akad. Nauk SSSR **135** (1960), 12–15 (Russian); translated as Soviet Math. Dokl. **1**, 1227–1230.

The author obtains various results concerning the limits of inverse and direct spectra. [The terminology is that of Raĭkov, Trudy Moskov. Mat. Obšč. **7** (1958), 413–438; MR **20** #6642.] For example, theorem 3 gives necessary and sufficient conditions for the limit of a nondegenerate inverse spectrum to be [fully, hyper-] complete. He also shows, by means of an example, that the product of a sequence of spaces complete in one of the above senses may fail to be complete in that sense, disproving thereby a conjecture of Kelley [Mich. Math. J. **5** (1958), 235–246; MR **21** #2173]. Similarly, direct sums may fail to preserve completeness properties.

J. G. Wendel (Ann Arbor, Mich.)

4933:

Vladimirov, D. A. On the completeness of a partially ordered space. Uspehi Mat. Nauk **15** (1960), no. 2 (92), 165–172. (Russian)

Kantorovič, Vulih, and Pinsker [*Funkcional'nyĭ analiz v poluuporyadočennyh prostranstvah*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1950; MR **12**, 340] defined a K-space as a general type of vector lattice in which bounded sets have a least upper bound. The paper under review examines the question of completeness of such spaces, and shows (with the continuum hypothesis) that completeness need not hold with respect to the (o)- and (t)-topologies. [For terminology, see the above reference.]

R. M. Baer (Berkeley, Calif.)

4934:

Singer, Ivan. On best approximation of continuous functions. Math. Ann. **140** (1960), 165–168.

Let $C(T)$ be the Banach space of all continuous real-valued functions on a compact Hausdorff space T. Let G be a linear subspace of $C(T)$ and $f \in C(T)$. It is said that there is a best approximation for f by means of G if, for some $g \in G$, $\|f-g\| = \text{dist}\,(f, G)$. The author shows that,

given an arbitrary linear subspace G, there exists $f \in C(T)$ with non-unique best approximation by means of G if and only if there exist two disjoint closed subsets F^+ and F^- of T and a Radon measure μ on T such that (1) $|\mu|(T)=1$, (2) μ is non-decreasing on F^+, non-increasing on F^- and $F=F^+ \cup F^-$ contains the carrier $S(\mu)$ of μ, (3) $\int g(t)d\mu(t)=0$ for all $g \in G$ where the integral is taken over F, and (4) there exists non-zero $g_0 \in G$ vanishing on $S(\mu)$. The special case when G is finite-dimensional was given by Haar [Math. Ann. **78** (1918), 294–311].

B. Yood (Eugene, Ore.)

4935:

Bessaga, C. A note on universal Banach spaces of a finite dimension. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 97–101.

A negative solution is given to the following problem of S. Mazur [Problem 41, "The Scottish Book", Los Alamos (mimeographed)]: Does there exist a finite-dimensional Banach space universal for (i.e., containing isometric images of) all two-dimensional Banach spaces? The solution results from the stronger proposition: No n-dimensional Banach space is universal for all those two-dimensional Banach spaces which have $(2n+2)$-gons as unit cells. A proof of the last statement is outlined. [Some recent papers dealing with closely related problems are Z. A. Melzak, Proc. Amer. Math. Soc. **9** (1958), 729–734; Canad. Math. Bull. **2** (1959), 31–32; MR **20** #4237; **21** #320, and B. Grünbaum, Bull. Res. Council Israel **7F** (1957/58), 133–135; MR **21** #4347.]

B. Grünbaum (Princeton, N.J.)

4936:

Hahn, Frank. Isolated points in a Banach space. Amer. Math. Monthly **67** (1960), 570–571.

Let B be a separable real Banach space, M a subset of B, and x a fixed point. The author shows, by an ingenious argument, that for each $\rho>0$, with at most a countable number of exceptions, every point of M whose distance from x does not exceed ρ can be approximated by points of M whose distances from x are strictly less than ρ. He also shows, by means of a counterexample, that one cannot draw the same conclusion if B is not separable.

{Reviewer's comments: (i) the page number in the footnote on page 570 should be 260; (ii) the author omits to mention that $S(x, \rho)$ denotes the closed ball (or sphere), and $S^0(x, \rho)$ the open ball, with centre x and radius ρ; (iii) there is a misprint in equation (1) on page 570, which should read $S(0, \rho) \cap M - S^0(0, \rho) \cap M = \emptyset$.}

A. F. Ruston (Sheffield)

4937:

Grünbaum, B. Projection constants. Trans. Amer. Math. Soc. **95** (1960), 451–465.

A normed linear space X is said to have the property P_s if for each normed linear space Y which contains X there exists a projection of bound not exceeding s of Y on X. If λ denotes the inf of the s for which X has property P_s, then X is said to belong to $\mathscr{P}_\lambda$. It is a consequence of a result of Goodner [same Trans. **69** (1950), 89–108; MR **12**, 266] that if X can be imbedded in a Y of class $\mathscr{P}_1$, then the corresponding λ for X is the inf of the bounds of the projections of Y on X. On the basis of this result, the λ for certain finite dimensional spaces is determined. These include the n-dimensional Euclidean and Minkowski spaces.

F. J. Murray (New York)

4938:

Orlicz, W. Contribution to the theory of Saks spaces. Fund. Math. **44** (1957), 270–294.

Let X be a linear space in which are introduced two norms: the fundamental norm $\|\ \|$ and the "starred" norm $\|\ \|^*$. If $d(x_n, x_0)=\|x_n-x_0\|^* \to 0$ and $\|x_n\|=O(1)$, the sequence $\{x_n\}$ is called 'w-convergent' ($x_n \to x_0$). In the set $X_s=E_x\{\|x\| \leq 1\}$, introduce the metric $d(x_1, x_2)=\|x_1-x_2\|^*$. The resulting metric space is denoted $X_s(\omega)$. If $X_s(\omega)$ is complete, it is called a 'Saks space'. Here in general X need not be complete with respect to either of the norms $\|\ \|$ and $\|\ \|^*$.

In the first part of the paper are given some supplements to earlier results [Orlicz, Studia Math. **15** (1955), 1–25; MR **17**, 511]. Here are some theorems from this part: If $X_s(\omega)$ is a Saks space, X is complete in the norm $\|x\|_0=\max(\|x\|, \|x\|^*)$. A distributive operator U mapping X into the normed linear space Y is called '$(X_s(\omega), Y)$-linear' if, for x_n and $x \in X_s(\omega)$, from $x_n \to x$ follows $U(x_n) \to U(x)$. Every linear functional ξ on $X_s(\omega)$ may be represented as $\xi=\xi_1+\xi_2$, where ξ_1 and ξ_2 are linear functionals on the normed spaces $(X, \|\ \|)$ and $(X, \|\ \|^*)$ respectively; furthermore $\|\xi\|_0=\|\xi_1\|+\|\xi_2\|^*$. Let U_n be a sequence of $(X_s(\omega), Y)$-linear operators. The sequence $\|U_n(x)\|$ is bounded for every $x \in X$, if and only if $\|U_n(x)\| \leq k\|x\|_0$.

The second part takes up linear functionals on the Saks space M_s^α of functions bounded on the line and summable to the power $\alpha \geq 1$, with the norms

$$\|x\| = \sup|x(t)|, \quad \|x\|^* = \left\{\int_{-\infty}^{\infty} |x(t)|^\alpha\, dt\right\}^{1/\alpha}.$$

In the third part the general theorems are applied to establishing the continuity of distributive operators on various function spaces.

M. I. Kadec (RŽMat **1959** #1676)

4939:

Lions, Jacques-Louis. Un théorème de traces; applications. C. R. Acad. Sci. Paris **249** (1959), 2259–2261.

To state the results requires the introduction of a multitude of different B-spaces. To start with, let E be a B-space of elements e, and let $\Lambda_1, \cdots, \Lambda_r$ be linear unbounded operators on E to E each of which generates a strongly continuous semi-group $G_j(t)$ such that $G_j(s)$ commutes with $G_k(t)$ for all j, k, s, t. Let $D(\Lambda_j)$ be the domain of Λ_j and set $E^1=\bigcap D(\Lambda_j)$. Then E^1 is a B-space under the norm $\|e\|+\sum 1^r \|\Lambda_j e\|$. Generally, let E^k be the subset of E^{k-1} such that $\Lambda^q e \in D(\Lambda_j)$ for each j and q. Here $\Lambda^q=\Lambda_1^{q_1}\Lambda_2^{q_2}\cdots\Lambda_r^{q_r}$, $|q|=q_1+q_2+\cdots+q_r=k-1$. The E^k-norm is defined to differ from the E^{k-1}-norm by the sum $\sum_{|q|=k}\|\Lambda^q e\|$. Here $E^0=E$ and each E^k is dense in E. Next, let $1<p<\infty$, $-1/p<\alpha<1-1/p$. Let $L_\alpha^p(0, \infty; X)$ be the set of strongly measurable functions $f(t)$ with values in X such that $\|t^\alpha f(t)\|^p \in L(0, \infty)$. Further, let $W^m(p, \alpha; \Lambda_j, E)=W^m$ be the set of functions $u(t)$ such that $D_t^j u \in L_\alpha^p(0, \infty; E^{m-j})$, $j=0, \cdots, m$. This is a B-space under the natural norm.

Finally the trace space T^k is introduced. Here $T^0=T^0(p, \alpha; \Lambda_j, E)$ is the set of elements of E such that

$$\|e\|_{T^0} = \|e\|+\Big\{\sum_{j=1}^{r}\int_0^\infty t^{(\alpha-1)p}\|G_j(t)e-e\|^p\, dt\Big\}^{1/p} < +\infty.$$

Here $E^1 \subset T^0 \subset E^0$. Similarly, T^k is the subset of E^k such that $\Lambda^q e \in T^0$ where $|q| = k$ and the norm differs from the E^k-norm by $\sum_{|q|=k} \|\Lambda^q e\|_{T^0}$.

The main result is now that if $u \in W^m$, then it is possible to define $u^{(k)}(0)$, $0 \leq k \leq m-1$ so that $u^{(k)}(0) \in T^{m-k-1}$. The linear continuous mapping

$$u \to \{u(0), u'(0), \cdots, u^{(m-1)}(0)\}$$

of W^m on $T^{m-1} \times T^{m-2} \times \cdots \times T^0$ is onto.

The applications include a number of interpolation theorems. Let $\tilde{E}$ be a second B-space with operators $\tilde{\Lambda}_j$ satisfying the above conditions and define the corresponding spaces $\tilde{E}^k$ and $\tilde{T}^k$. If π defines a linear bounded mapping of E^k into $\tilde{E}^k$ for $k = 0, \cdots, m$, if p and α satisfy the stated conditions, then π defines a linear bounded mapping of T^k into $\tilde{T}^k$, $0 < k < m$. For special choices of E results of Sobolev, Gagliardo, Slobodeckiĭ, Vacherin, and Peetre are obtained. *E. Hille* (New Haven, Conn.)

4940:

Lions, Jacques-Louis. Sur certains théorèmes d'interpolation. C. R. Acad. Sci. Paris **250** (1960), 2104–2106.

This is a continuation and development of the ideas presented in the author's note [see preceding review]. Two general methods for the construction of interpolation theorems are given. In view of the notational complications the author restricts himself to the case of a single operator Λ which generates a strongly continuous semigroup $G(t)$ on E and $T^0(p, \alpha)$ is the subset of E such that $t^{\alpha-1}[G(t)e - e] \in L^p(0, \infty; E)$, $0 < \theta \equiv \alpha + 1/p < 1$. Let X be a locally compact space, $d\mu$ a positive measure, ψ_0 and ψ_1 positive measurable functions such that $(\psi_0)^{b_0}$ and $(\psi_1)^{b_1}$ are locally integrable. Let $L^b(X, \psi)$ be the space of functions f such that $\psi f \in L^b(X)$.

Now if π is a linear bounded operator both on $E' = D(\Lambda)$ to $L^{b_0}(X, \psi_0)$ and on E to $L^{b_1}(X, \psi_1)$, then if $1 < p \leq \inf(b_0, b_1) \leq \infty$, π is a bounded operator on $T^0(p, \alpha)$ to $L^b(X, \psi_0^{1-\theta}\psi_1^{\theta})$ where $1/\beta = (1-\theta)/b_0 + \theta/b_1$, while if $p > \inf(b_0, b_1)$ then $T^0(p, \alpha)$ maps continuously into an intersection of such spaces.

A similar result holds if Λ is the infinitesimal generator of a bounded group $G(t)$. Let $S_0(p, \alpha)$ be the subset of E such that $t^{\alpha-2}[G(t)e + G(-t)e - 2e] \in L^p(0, \infty; E)$. If now π is a linear bounded operator both on $D(\Lambda^2)$ into $L^{b_0}(X, \psi_0)$ and on E into $L^{b_1}(X, \psi_1)$ and if $1 < p \leq \inf(b_0, b_1) \leq \infty$, then π is bounded on $S_0(p, \alpha)$ into $L^b(X, \psi_0^{1-\theta/2}\psi_1^{\theta/2})$ where $1/b = (1-\theta/2)/b_0 + (\theta/2)/b_1$, etc.

The method gives the results of M. Riesz in Acta Math. **49** (1926), 465–497, except for the convexity of the norm, as well as other classical results.

For the second method, the author considers functions φ defined on R^n satisfying condition (P_k): $\varphi > 0$, φ and $1/\varphi$ have derivatives of all orders, $D^p\varphi/\varphi \in L^\infty(R^n)$ for $|p| \leq k$. Let $H^{k,a}$ be the class of functions f such that $\mathcal{F}^{-1}[(1+\xi^2)^{k/2}\mathcal{F}f] \in L^a$, where $\mathcal{F}$ is the Fourier transformation. Let $H_\varphi^{k,a}$ be the set of functions f such that $\varphi f \in H^{k,a}$, k positive integer. If φ_0 and φ_1 satisfy (P_k), and if π is a linear bounded operator both on $H_{\varphi_0}^{k,a}$ into $L^{b_0}(X, \psi_0)$ and on $H_{\varphi_1}^{k,a}$ into $L^{b_1}(X, \psi_1)$ with $1 < a \leq \inf(b_0, b_1)$, then π is also bounded on $H_\varphi^{k,a}$ into $L^b(X, \psi)$, where $\varphi = \varphi_0^{1-\theta}\varphi_1^{\theta}$, $\psi = \psi_0^{1-\theta}\psi_1^{\theta}$ and β is defined above.

Proofs are sketched. *E. Hille* (New Haven, Conn.)

4941:

Anderson, F. W. A class of function algebras. Canad. J. Math. **12** (1960), 353–362.

$C(X)$ is characterized as a vector lattice, as an l-ring, and as an algebra over the reals, for the special case in which X is a P-space (i.e., $C(X)$ is a regular ring). The conditions are too involved to reproduce here.

L. Gillman (Rochester, N.Y.)

4942:

Gillman, Leonard; Kohls, Carl W. Convex and pseudoprime ideals in rings of continuous functions. Math. Z. **72** (1959/60), 399–409.

This is an addition to the growing literature on the linear-algebraic properties of the linear algebra $\mathcal{C}(X)$, X a completely regular topological space. A few definitions and theorems follow. An ideal I is convex if $0 \leq f \leq g \in I$ implies $f \in I$; it is absolutely convex if $|f| \leq |g|$ and $g \in I$ implies $f \in I$; it is pseudoprime if f, $g \notin I$ implies $fg \neq 0$. The following are equivalent: I is pseudoprime, I contains a prime ideal, if $J \supset I$ is convex then $\mathcal{C}(X)/J$ is totally ordered, the convex ideals $J \supset I$ form a chain, the prime ideals $J \supset I$ form a chain. An absolutely convex ideal is the intersection of pseudoprime absolutely convex ideals.

R. Arens (Los Angeles, Calif.)

4943:

Fréchet, Maurice. L'espace des courbes n'est pas un espace de Banach. C. R. Acad. Sci. Paris **250** (1960), 2787–2790.

It is shown that if certain natural definitions of distance, scalar product, zero element and norm are assigned to the set of continuous curves in R_3, then it is impossible to choose a definition of addition so that a Banach space is obtained. This solves a problem discussed earlier by the author [Colloq. Math. **6** (1958), 33–40; **7** (1959), 201–204; MR **20** #7208; **22** #2887].

C. W. Kohls (Rochester, N.Y.)

4944:

Fréchet, Maurice. Simplification d'une démonstration donnée dans une note précédente. C. R. Acad. Sci. Paris **251** (1960), 9.

This note gives a shorter, more direct proof of the result stated in the preceding review.

C. W. Kohls (Rochester, N.Y.)

4945:

Bishop, Errett; de Leeuw, Karel. The representations of linear functionals by measures on sets of extreme points. Ann. Inst. Fourier. Grenoble **9** (1959), 305–331.

Suppose X is a compact Hausdorff space and $C_r(X)$ and $C_c(X)$ the spaces of real and complex continuous functions on X, respectively, supplied with the usual supremum norms. Let H be the collection of non-negative Baire measures on X, and B a subspace of C_r or C_c that contains the constant functions. $H_x(B)$ shall be the subset of H for which (*) $f(x) = \int f d\mu$ for every $f \in B$. If S is any subset of X define $i_B(S) = \{y : f(y) = f(x) \text{ for some } x \in S \text{ and all } f \in B\}$. The Choquet boundary of B, $M(B)$, is the set of $x \in X$ such that any $\mu \in H_x(B)$ satisfies $\hat{\mu}(i_B(\{x\})) = 1$, where $\hat{\mu}$ is the regular Borel extension of μ. In case B separates points of X, then the Choquet boundary of B is the set of x for which the only measure in $H_x(B)$ which satisfies (*) is the unit point mass at x.

If B is a separable subspace of $C_r(X)$ it turns out that $M(B)$ is a G_δ. Further, if $\eta \in H$, then there exists a $\mu \in H$, concentrated on $M(B)$, such that $\int f d\eta = \int f d\mu$ for all $f \in B$. From this the following result is an immediate consequence: If B is a separable subspace of $C_r(X)$ or $C_c(X)$, then any $L \in B^*$ has a representation $L(f) = \int f d\mu$, with μ a Baire measure concentrated on $M(B)$. In turn, this latter result leads immediately to the following theorem of Choquet: Let X be a compact, convex, metrizable subset of a locally convex real linear topological space E. If X_e is the set of extreme points of X, then X_e is a G_δ in X and every $x \in X$ has a representation $x = \int y d\mu(y)$ for $\mu \in H$ and $\mu(X_e) = \mu(X) = 1$.

In case B is not separable but distinguishes points of X, then the representation theorem takes the following form. Let Σ be the σ-ring generated by $M(B)$ and the Baire sets of X. Then any $L \in B^*$ has a representation in the form $L(f) = \int f d\mu$, where μ is a measure of Σ such that $\mu(T) = 0$ for each $T \in \Sigma$ disjoint from $M(B)$. Using this result a generalized Choquet theorem can be stated in the obvious way in case the convex compact set X is not necessarily metrizable.

A subset Y of X is called a boundary for B if for each $f \in B$ there is some $y \in Y$ such that $f(y) = \|f\|$. If A is a uniformly closed subalgebra of $C_c(X)$ that distinguishes points and contains the constants then the Šilov boundary is the smallest closed boundary for A. The authors prove here that if every $x \in X$ is a G_δ then any boundary for A contains the Choquet boundary, and in the general case any boundary for A which is a Baire set contains the Choquet boundary.

The paper ends with some examples which show that the Choquet boundary, which is a G_δ in the separable case, can be arbitrarily bad in general.

A. Devinatz (Princeton, N.J.)

4946:

Wloka, Josef. Distributionen und Operatoren. Math. Ann. **140** (1960), 227–244.

Section I discusses the embedding of the distributions of $\mathfrak{D}_+'$, i.e., those with supports on a right halfline, in the field M of Mikusiński operators, by means of the relation $D \to f/\varphi$, whenever $D * \varphi = f$, where $D \in \mathfrak{D}_+'$, $\varphi \in \mathfrak{D}$, and so f is infinitely differentiable with support on a right halfline; it is shown that the mapping is one-one. (The notation for distribution theory and convolutions, here and in the article, is that of Schwartz, *Théorie des distributions*, I and II [Actualités Sci. Ind., no. 1091, 1122, Hermann, Paris, 1950, 1951; MR **12**, 31, 833], and, for operator theory, that of Mikusiński, *Operational calculus* [Pergamon, New York, 1959; MR **21** #4333].) Any $D \in \mathfrak{D}_+'$ has the form $D = \sum_{i=0}^{\infty} \delta^{(n_i)}(t) * \delta(t-i) * F_i(t)$, where the $F_i(t)$ are continuous functions with the same halfline as support and the sum is locally finite: consequently the operations corresponding to distributions are those of the form $\sum_{i=0}^{\infty} s^{n_i} h^i F_i$ {the formula I.(8) of the paper has an obvious misprint}.

Section II generalizes the theory of operator functions of a variable λ to operator distributions. Let Φ_0 be the class of all continuous functions of t, Φ the set of all distributions in λ of form $\partial^k F(t, \lambda)/\partial\lambda^k$ for some integer k. Composition (in t) of an element of Φ by a continuous $a(t)$ is defined by $\partial^k F(t, \lambda)/\partial\lambda^k * a(t) = \partial^k\{F(t, \lambda) * a(t)\}/\partial\lambda^k$. Null-divisors do not exist in this multiplication, and Φ is closed for composition, addition and differentiation with respect to λ. The quotient class MD of elements of the form $f(t, \lambda)/a(t)$, $f(t, \lambda) \in \Phi$ and $a(t)$ continuous, is called the class of operator distributions (in λ). Rules for commutation of differentiation and composition (with respect to t, as throughout) are proved, and definitions of composition with infinitely differentiable operator functions and a convergence are defined.

Section III is concerned with differential equations $\sum a_i \partial^i x/\partial\lambda^i = f$, where a_i are Mikusiński operators (in t) and x and f are in MD. It is shown that the theory reduces to that in the Mikusiński class M: the classification of operators into pure, logarithmic and mixed is unaffected by the greater generality of solutions, all solutions of the homogeneous equation being in M.

J. L. B. Cooper (Cardiff)

4947:

Bouix, Maurice. Condition pour qu'une série de fonctions orthogonales continues représente une distribution. C. R. Acad. Sci. Paris **251** (1960), 316–317.

The following generalization of the conditions for convergence of a trigonometrical series to a distribution is stated without proof. Let $u_k(x)$ be an orthonormal sequence of continuous functions on (a, b), with $u_1(x)$ a constant and $u_i(a) = u_i(b)$, $\int_a^b u_i(x)\,dx = 0$ for $i > 1$; let $u_i'(x) = \sum a_{ik} u_k(x)$, and $\gamma_{jk^n} = \sum a_{ji_1} a_{i_1 i_2} \cdots a_{i_{n-1} k}$. Then a necessary and sufficient condition that $\sum_1^\infty b_k u_k(x)$ represent a distribution of order $\leq p+1$ is that for some M,

$$b_k^2 \leq M^2 \sum_j (\gamma_{jk^p})^2.$$

J. L. B. Cooper (Cardiff)

4948:

Hirata, Yukio. On convolutions in the theory of distributions. J. Sci. Hiroshima Univ. Ser. A **22**, 89–98 (1958).

A locally convex space E such that (i) $\mathscr{D} \subset E \subset \mathscr{D}'$ and the injections $\mathscr{D} \to E \to \mathscr{D}'$ are continuous, (ii) $\mathscr{D}$ is dense in E, where the notation of distribution theory is used, is called 'admissible'. Elements of E are identified with distributions. E is 'permitted' if, in addition, for any $T \in E$, $(\alpha_k \cdot T) * \rho_k$ and $\alpha_k \cdot (T * \rho_k)$ tend to T as $k \to \infty$, if (α_k) is a sequence of multiplicators and (ρ_k) of regularisations. The 'c-dual' E^* of E is the set of distributions T such that $T * \xi$ exists for every ξ in E. Conditions for distributions to be in E^*, for spaces formed from E by successive formation of dual and c-dual to be admissible and permitted, and conditions for E and E^{**} to form algebras are discussed.

J. L. B. Cooper (Cardiff)

4949:

Shiraishi, Risai. On the definition of convolutions for distributions. J. Sci. Hiroshima Univ. Ser. A **23**, 19–32 (1959).

The usual notation of the theory of distributions is used. Let $T, S \in (\mathscr{D}')_x$; the author proves the equivalence of the following three (Schwartz's and Chevalley's) definitions of the convolution $T * S$: (1) $(S * T, \varphi) = \iint (S_x \otimes T_y)\varphi(x+y)\,dxdy$; (2) $(S * T, \varphi) = \int S(\check{T} * \varphi)\,dx$; (3) $(S * T, \varphi * \psi) = \int (S * \check{\varphi})(\check{T} * \psi)\,dx$; provided that the integrand belong (1) to $(\mathscr{D}_{L'}')_{x,y}$ for any $\varphi \in \mathscr{D}$, (2) to $(\mathscr{D}_{L'}')_x$ for any $\varphi \in \mathscr{D}$, (3) to L^1 for any $\varphi, \psi \in \mathscr{D}$, respectively. The conditions on the integrand remain equivalent after replacing $\mathscr{D}$ by $\mathscr{S}$, and then they imply that S, T

and $S * T$ belong to $\mathscr{S}'$ (unless S or T vanishes identically). The corresponding (to (1) and (3)) definitions of simultaneous convolution $S * T * U$ are equivalent. Some consequences concerning the spaces $H_c'^m$ [cf. L. Schwartz, J. Analyse Math. 4 (1954/55), 88–148; MR 18, 220] are given. *S. Łojasiewicz* (Kraków)

4950:

Braga, Carmen Lys Ribeiro; Schönberg, Mario. Formal series and distributions. An. Acad. Brasil. Ci. **31** (1959), 333–360.

In this paper is discussed the generalization of the notion of a function by algebraic means, using linear sequence spaces. Coordinates are introduced into the space $L_2(R^n)$ of square integrable functions in n-dimensional space R^n by an orthonormal set $\{\phi_n\}$, and the completion E of the resulting sequence space l_2 in the weak (direct product) topology is called the space of generalized functions. When the orthonormal set is composed of Hermite functions, any polynomial of differentiation may be replaced by a matrix of a finite number of rows, and so is defined on the whole of E. Similarly the Fourier transform may be defined for every element of E. E contains the space $\mathscr{S}'$ of slowly increasing distributions of Schwartz and also elements not contained in the space $\mathscr{D}'$ of distributions of Schwartz. E is extended to $\mathscr{E} = \sum_{n=1}^{\infty} \oplus (\otimes E)^n$, and generalized functions are considered as classes of $\mathscr{E}$ under the equivalence relation $\sum_r (c_r - c_r') = 0$, where $c = \{c_r\}$, c_r belongs to $(\otimes E)^n$, $r = (r_1, \cdots, r_n)$ with integers $r_1, \cdots, r_n$. {It would seem necessary to the reviewer that this equivalence relation is ambiguous unless the series is required to be absolutely summable to zero.} This extension is necessary to consider simultaneously formal series of generalized functions which arise from different orthonormal sets, and also to obtain a space $\mathscr{E}$ which contains $\mathscr{D}'$. The operators of differentiation and Fourier transformation can be extended to the whole of $\mathscr{E}$. No applications of this approach are discussed.

John G. Taylor (Orsay)

4951:

Sato, Mikio. Theory of hyperfunctions. I. J. Fac. Sci. Univ. Tokyo. Sect. I **8** (1959), 139–193.

An exposition of the elements of a theory of hyperfunctions (analytic distributions), motivated by the classical work of L. Fantappiè on analytic functionals and of L. Schwartz on the theory of distributions, and by a relatively recent article by G. Köthe putting together these viewpoints [Math. Z. **57** (1952), 13–33; MR **14**, 563]. A summary appeared in two notes by the author [Proc. Japan Acad. **34** (1958), 126–130, 604–608; MR **20** #2618; **21** #5894]. The present article is restricted to the case of one variable. Let $\mathbf{C}$ be the complex field, $\mathbf{R}$ the real axis in $\mathbf{C}$, S a locally closed subset of $\mathbf{R}$, $\mathfrak{D}(S)$ the set of all open subsets of $\mathbf{C}$ which contain S as a closed subset and $\mathfrak{U}(D)$ the algebra of all holomorphic complex functions in an open subset D of $\mathbf{C}$. Given S, consider the set of all couples (φ, D) with $\varphi \in \mathfrak{U}(D-S)$, $D \in \mathfrak{D}(S)$, and introduce an equivalence relation in this set by considering (φ_1, D_1) and (φ_2, D_2) as equivalent if there are $D \in \mathfrak{D}(S)$, $D \subset D_1 \cap D_2$, $f \in \mathfrak{U}(D)$ such that $\varphi_2(z) - \varphi_1(z) = f(z)$ for $z \in D - S$. Each such equivalence class is a hyperfunction on S, which, in a certain sense, can be regarded as "boundary values" of an analytic function. For such hyperfunctions, a study is made of holomorphy, restriction, change of variables, derivatives and indefinite integrals, carriers, integration, localizability, linear differential equations, and examples dealing with Dirac's δ-function, Heaviside's Y-function, power function, etc. An alternative way of defining hyperfunctions in terms of harmonic functions instead of analytic functions is pointed out. Analytic distributions on Riemann surfaces are only very briefly mentioned at the end of the exposition.

L. Nachbin (Waltham, Mass.)

4952:

Rosenblum, Marvin. The absolute continuity of Toeplitz's matrices. Pacific J. Math. **10** (1960), 987–996.

Let $W \in L^2(-\pi, \pi)$. The associated Toeplitz matrix is $T_0 = (w_{j-k})$ $(j, k = 0, 1, 2, \cdots)$ with

$$w_n = (2\pi)^{-1} \int_{-\pi}^{\pi} W(\phi) \exp(-in\phi)\, d\phi \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

If W is real-valued and semi-bounded, T_0 defines a semi-bounded symmetric operator in l^2, whose Friedrichs extension is the Toeplitz operator $T(W)$. The following theorems are proved. (1) $T(W)$ is absolutely continuous (that is, the associated spectral measure is Lebesgue absolutely continuous). This strengthens previous results by P. Hartman and A. Wintner [Amer. J. Math. **72** (1950), 359–366; **76** (1954), 867–882; MR **12**, 187; **17**, 499] and C. R. Putnam [Trans. Amer. Math. Soc. **87** (1958), 513–525; MR **20** #6659]. (2) If W is even and $\sum n|w_n| < \infty$, then $T(W)$ is unitarily equivalent to the multiplication operator $f \to Wf$ on a subspace $L^2(A)$ where A is a suitable subset of $(0, \pi)$. In the proof of (1) the author depends on the Aronszajn-Donoghue theory of exponential representations of holomorphic functions, while for (2) a theorem of the reviewer on perturbation of continuous spectra is used. *T. Kato* (Tokyo)

4953:

Taam, C. T. Compact linear transformations. Proc. Amer. Math. Soc. **11** (1960), 39–42.

Let E be a compact Hausdorff space and let $C(E)$ denote the Banach algebra of all complex-valued continuous functions on E with the uniform topology. The author shows that any compact linear transformation from an arbitrary Banach space X to $C(E)$ can be approximated arbitrarily closely in norm by bounded linear transformations on X to $C(E)$ whose ranges are finite dimensional. *R. S. Phillips* (Stanford, Calif.)

4954:

Thorp, Edward O. Note on linear operators. J. Reine Angew. Math. **203** (1960), 110–112.

If X and Y are Banach spaces, let $[X, Y]$ be the space of all bounded linear operators from X into Y. If Y is inseparable, under what conditions will all operators in $[X, Y]$ be compact? Using theorems known for separable spaces, the author shows that this is true in the following cases: $[l^p(S), l^q(T)]$ for $\infty > p > q \geq 1$; $[c_0(S), l^q(T)]$, $[c(S), l^q(T)]$, $\infty > q \geq 1$; $[X, l^1(T)]$, where X is either quasi-reflexive or weakly complete or has a separable conjugate space. Further, for S and T uncountable and $\infty > p > q > 1$,

every $A \in [l^p(S), l^q(T)]$ has a nondense range and nonzero kernel, and is compact. *V. Ptàk* (Prague)

4955:

Thorp, E. O. Projections onto the subspace of compact operators. Pacific J. Math. **10** (1960), 693–696.

Let U and V be Banach spaces. Let $[U, V]$ be the space of bounded operators on U into V. The paper concerns the problem of existence of the closed linear complement of the space of compact (completely continuous) operators in $[U, V]$. Suppose U has a closed subspace X and V has a closed subset Y, each of which has a closed complement. Then the set of compact operators has no closed complement in the following cases: (1) X is isomorphic to l^p, $1 \leq p < \infty$, Y is isomorphic to l^q, $1 \leq p \leq q \leq \infty$ or c_0 or c; (2) X is isomorphic to c_0; Y is isomorphic to l^∞, c_0 or c; (3) X is isomorphic to c; Y is isomorphic to l^∞. *P. Saworotnow* (Washington, D.C.)

4956:

Nikodým, Otton Martin. Contribution to the theory of maximal, normal operators in a separable and complete Hilbert-Hermite-space. II. An item of operational calculus and its application to the resolvent and spectrum. J. Reine Angew. Math. **203** (1960), 90–109.

In this paper the author gets a nonstandard classification of the spectrum of normal operators in Hilbert space, and also develops the operational calculus. Unfortunately, without thorough control of the author's special nomenclature developed in his earlier papers [MR **15**, 803; **20** #1224], the present paper is virtually unreadable. *F. H. Brownell* (Seattle, Wash.)

4957:

Krabbe, Gregers L. Normal operators on the Banach space $L^p(-\infty, \infty)$. II. Unbounded transformations. Bull. Amer. Math. Soc. **66** (1960), 86–90.

[For part I see same Bull. **65** (1959), 270–272; MR **21** #5896.]

Let us denote by L^+, the intersection of the family $\{L^p(R); 1 < p < \infty\}$, $R = (-\infty, \infty)$; by S, the set of all operators K on L^+ such that $\|K\|_q < +\infty$ whenever $1 < q < \infty$. If f is a function on R, then $[\hat{}(f)]_p$ is the endomorphism of L^p whose restriction T to L^+ belongs to S and such that (Fourier transform of Tx) = (Fourier transform of x)$\cdot f$ for all $x \in L^p$; $p(f)$ is the class of all operators T with domain $D(T)$ and range in L^p, which commute with each member of the family $\{[\hat{}(\phi_\alpha)]_p; \alpha > 0\}$ and are such that $[(\phi_\alpha)]_p Tx = [\hat{}(f\phi_\alpha)]_p x$ for all $x \in D(T)$ and all $\alpha > 0$, where ϕ_α is the characteristic function of the interval $(-\alpha, \alpha)$; D_p is the transformation $D_p x =$ derivative of x, defined on the set $D(D_p)$ of all locally absolutely continuous $x \in L^p$, and $P_p = (i/2\pi)D_p$. The following main results are stated. (1) If g is a real-valued piecewise monotone and continuous function on R such that $|g(-\infty)| = |g(+\infty)| = \infty$ and if $Q \in p(g)$, then there exists a spectral resolution E^Q in L^p such that $Q \subseteq \int \lambda E^Q(d\lambda)$ and $I_p \subseteq \int 1 \cdot E^Q(d\lambda)$, where $I_p x = x$ for all $x \in L^p$; if, in addition, there exists a continuous function f on R such that the composition function $\phi = f \circ g$ is of locally bounded variation, and if $T \in p(\phi)$, then $T \subseteq \int f(\lambda) E^Q(d\lambda)$. (2) If g is a complex-valued function of locally bounded variation on R and if $T \in p(g)$ (in particular, if $T = [\hat{}(g)]_p$), then $T \subseteq \int g(\lambda) E^Q(d\lambda)$, where $Q = P_p$. The relation $S \subseteq \int h(\lambda) E(d\lambda)$ means that for all $x \in D(S)$ the Riemann-Stieltjes integral $\int_{-\infty}^{\infty} h(\lambda) E(d\lambda) x$ converges to Sx (in the topology of L^p). *N. Dinculeanu* (Bucharest)

4958:

Nelson, Edward. Correction to "Kernel functions and eigenfunction expansions". Duke Math. J. **26** (1959), 697–698.

The paper corrected was in same J. **25** (1958), 15–27 [MR **19**, 969].

4959:

Maurin, K. Spektraldarstellung der Kerne. Eine Verallgemeinerung der Sätze von Källén-Lehmann und Herglotz-Bochner u.a. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 461–470. (Russian summary, unbound insert)

The author studies the spectral resolution of positive-definite operators which have kernels in the sense of L. Schwartz, obtaining eigenkernels which for almost every λ are again positive-definite kernels. If the operator has certain group invariance properties so do the eigenkernels. *E. Nelson* (Princeton, N.J.)

4960:

Maurin, K. Allgemeine Eigenfunktionsentwicklungen. Spektraldarstellung abstrakter Kerne. Eine Verallgemeinerung der Distributionen auf Lie'schen Gruppen. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 471–479. (Russian summary, unbound insert)

The author gives an abstract treatment of eigenfunction expansions and spectral resolution of self-adjoint operators. The discussion is based on the assumption of a dense nuclear subspace ϕ of a Hilbert space H (theorem 3 constructs such a space ϕ having the desired properties). Eigenfunctions are then elements of ϕ' and the operators are expressed as direct integrals of operators with kernels in $\phi' \otimes \phi'$. The main result is the lemma in section 1: Let ϕ be a locally convex subspace of the Hilbert space H, let $F: H \to \hat{H} = \int \hat{H}(\lambda)\, d\mu(\lambda)$ express H as a direct integral of Hilbert spaces, and suppose that the restriction of F to ϕ is nuclear. Then for almost every λ, the map $\varphi \to F\varphi(\lambda)$ is continuous and nuclear. *E. Nelson* (Princeton, N.J.)

4961:

Porath, Günter. Störungstheorie der isolierten Eigenwerte für abgeschlossene lineare Transformationen im Banachschen Raum. Math. Nachr. **20** (1959), 175–230.

This is a dissertation dealing with the regular (analytic) perturbation of the spectrum of a linear operator in a Banach space. The results as well as the methods are straightforward generalizations of those of previous authors (Rellich, Sz.-Nagy, Schäfke, Schröder, Wolf and the reviewer). *T. Kato* (Tokyo)

4962:

Sebastião e Silva, José. Le calcul opérationnel pour des opérateurs à spectre non borné. Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I (8) **6** (1960), 3–13. (Italian summary)

This paper announces without proofs an operational

calculus which is applicable to certain types of operators with unbounded spectra. The ideas are similar to those in two earlier papers by the same author [Portugal. Math. **14** (1956), 105–132; **17** (1958), 1–17; **18** (1959), 154–155; MR **18**, 137; **21** #296], but the applications are to a somewhat different class of problems.

Let F be a nonvoid, closed, proper subset (in general unbounded) of the complex plane $\mathbf{C}$. Let F_k be the set of points of $\mathbf{C}$ whose distance from F is less than $1/k$, and let $\mathfrak{A}_k(F)$ be the normed space of complex functions φ which are holomorphic on F_k and such that

$$\|\varphi\|_k = \sup_{z\in F_k} \frac{|\varphi(z)|}{(1+|z|)^k} < +\infty.$$

With obvious identifications we have $\mathfrak{A}_k(F)\subset\mathfrak{A}_{k+1}(F)$, and the space $\mathfrak{A}_\omega(F)$ is defined as the inductive limit of the spaces $\mathfrak{A}_k(F)$.

If $\lambda\in\mathbf{C}-F$, the element of $\mathfrak{A}_\omega(F)$ whose value at z is $(z-\lambda)^{-1}$ is denoted by $\mathfrak{h}(\lambda)$. Certain assumptions are placed on F to insure that each element φ of $\mathfrak{A}_\omega(F)$ has a representation $\varphi=(2\pi i)^{-1}\int\varphi(\lambda)\mathfrak{h}(\lambda)\,d\lambda$, the integration being taken over the boundary of a suitable open set (depending on φ), and the integral being defined in a suitable generalized sense, with all limits understood in the topology of $\mathfrak{A}_\omega(F)$.

Next, let $\mathbf{A}$ be a locally convex topological algebra with unit $\mathbf{e}$ and with multiplication continuous in each factor separately. Let R be the class of elements $\mathbf{a}\in\mathbf{A}$ such that (1) $\mathbf{a}-\lambda\mathbf{e}$ is a regular element of $\mathbf{A}$ if $\lambda\notin F$, (2) for each k, $(\mathbf{a}-\lambda\mathbf{e})^{-1}$, as a function with values in $\mathbf{A}$, is bounded in each of the sets $\mathbf{C}-F_k$. Then R is in 1-1 correspondence with the family of all continuous homomorphisms $\mathfrak{H}$ of $\mathfrak{A}_\omega(F)$ into $\mathbf{A}$ such that $\mathfrak{H}(\varphi)=\mathbf{e}$ if $\varphi(z)\equiv 1$. The correspondence $\mathfrak{H}\leftrightarrow\mathbf{a}$ is exhibited by the formulas

$$\mathfrak{H}(z)=\mathbf{a},\quad \mathfrak{H}(\varphi)=(2\pi i)^{-1}\int\varphi(\lambda)(\mathbf{a}-\lambda\mathbf{e})^{-1}\,d\lambda,$$

where the sense of the integral is to be suitably understood. The operational calculus then arises by defining $\varphi(\mathbf{a})=\mathfrak{H}(\varphi)$.

The indicated applications include the solution of certain types of partial differential equation problems. A sketch of the basic idea runs something like this: Given a certain differential operator, defined on a subset of a certain locally convex space E, the space E and the operator are both extended so that we have to deal with a continuous linear operator defined on the whole extended space $\bar{E}$. This space, in certain cases, is a space of distributions, and the algebra A is the class of linear mappings of $\bar{E}$ into itself, continuous in a suitable sense. The author discusses two types of abstract Cauchy problems, and three examples connected with the wave equation and the equation of diffusion.

A. E. Taylor (Los Angeles, Calif.)

4963:

Combes, Jean. Sur certains systèmes infinis d'équations linéaires. I. Ann. Fac. Sci. Univ. Toulouse (4) **21** (1957), 255–265 (1959).

System (1) $\sum_{j=1}^{\infty}a_{ij}x_j=y_i$ $(i=1,2,\cdots)$ (and the corresponding matrix $A=((a_{ij}))$) is triangular if $a_{ij}=0$ for $j<i$, and is regular triangular (r.t.) if also $a_{ii}\neq 0$ (in which case it can be assumed that $a_{ii}=1$). Every r.t. A has a unique left inverse $B=((b_{ij}))$. B is also a right inverse and is r.t. Write (1) in the form $AX=Y$ where X, Y are matrices of one column. If $A=((a_{ij}))$, let $A'=((|a_{ij}|))$. Suppose r.t. system (1) has the solution X, and let $B'A'X'$ exist; then $X=BY$ holds (B=inverse of A indicated above), and $B'Y'$ exists. In particular, if $AX=0$ and if $B'A'X'$ exists, then $X=0$. Thus, if $AX=Y$ and $X\neq BY$, or if $AX=0$ and $X\neq 0$, then $B'A'X'$ does not exist; which is to say that at least one of the series $\sum_{j=1}^{\infty}|x_j|(\sum_{k=i}^{j}|b_{ik}a_{kj}|)$ $(i=1,2,\cdots)$ diverges (and for $AX=0$, $X\neq 0$, infinitely many of these diverge). The author expresses this by saying that if (1) has a solution $X\neq BY$, then X cannot be "too small". Let Y be "sufficiently small" so that $A'B'Y'$ exists; then $X=BY$ exists, and satisfies (1).

Other results are given, and applications concerning analytic functions are to appear in part II.

I. M. Sheffer (University Park, Pa.)

4964:

Tanabe, Hiroki. A class of the equations of evolution in a Banach space. Osaka Math. J. **11** (1959), 121–145.

The author constructs the fundamental solution by a method of successive approximations for the equation $du/dt=A(t)u+f(t)$ under the hypotheses: (1) for each $t\in[a,b]$, $A(t)$ is the infinitesimal generator of a semi-group of contraction operators on a Banach space X; (2) the domain of $A(t)$ is independent of t; (3) the operator $B(t,s)=[I-A(t)][I-A(s)]^{-1}$ is uniformly bounded for $a\leqq s,t\leqq b$; (4) $B(t,s)$ satisfies a Lipschitz condition in t for every s in the uniform operator topology; (5) $B(t,s)$ is strongly continuously differentiable in t for every s; (6) there exist positive constants C and t_0 such that $\|d/dt\exp(tA(s))\|\leqq C/t$ for any $s\in[a,b]$ and $t\leqq t_0$; and (7) $f(t)$ is Hölder continuous in $a\leqq t\leqq b$. T. Kato has previously established the existence of a fundamental solution $U(t,s)$ for this equation without the assumption (6) [J. Math. Soc. Japan **5** (1953), 208–234; MR **15**, 437], however the present construction is much more convenient for deducing various other properties of the fundamental solution. It is shown, for instance, that $\partial U(t,s)/\partial t=A(t)U(t,s)$ is a bounded operator with norm bounded by $C/(t-s)$ for some constant C. Under somewhat more restrictive hypotheses on the differentiability of $B(t,s)$ for fixed s, it is shown that higher derivatives of $U(t,s)$ exist. The author also develops a perturbation theory for the equation $du/dt=(A(t)+B(t))u+f(t)$ where the perturbing operator function $B(t)$ satisfies less restrictive conditions as a function of t than $A(t)$, but where $B(t)$ is in some sense dominated by $A(t)$ for each t. In the case of differential operators the perturbing operators correspond to lower order differential operators. Finally the theory is applied to a parabolic partial differential equation of considerable interest.

R. S. Phillips (Stanford, Calif.)

4965:

Karlin, Samuel. Positive operators. J. Math. Mech. **8** (1959), 907–937.

The author is concerned with operators T in a Banach space E that are positive in the sense of mapping a positive cone K into itself, and particularly with generalizations of the classical theorems of Perron and Frobenius on matrices with non-negative elements. The proofs mainly depend on exploiting theorems on power series with non-negative coefficients, notably the following theorem of Hardy and Littlewood [Proc. London Math. Soc. (2) **13**

(1914), 174–191]: If $f(\lambda)=\sum_1^\infty a_n\lambda^n$ has radius of convergence 1, if $a_n\geqq 0$, and if $f(\lambda)\sim a(1-\lambda)^{-k}$ as $\lambda\to 1-$, then $n^{-k}\Gamma(k+1)\sum_{i=1}^n a_i\to a$ as $n\to\infty$. It is assumed that K and K^* span E and E^*, and that T has non-zero spectral radius, which is normalized to be 1. Among many other results, it is proved that if 1 is a pole of order k of the resolvent of T, then $n^{-k}\Gamma(k+1)\sum_{i=1}^n T^i$ converges uniformly as $n\to\infty$ to $(T-I)^{k-1}P$, where P is a projection. This extends in a certain direction a theorem of Kreĭn and Rutman [Uspehi Mat. Nauk **3** (1948), no. 1 (23), 3–95; Amer. Math. Soc. Transl. no. 26 (1950); MR **10**, 256; **12**, 341] on the existence of positive eigenvectors of a compact positive operator. Since it depends on the spectral radius being a pole of the resolvent operator it does not appear to yield results of the kind obtained by H. Schaefer [Math. Ann. **138** (1959), 259–286; MR **21** #5135] and the reviewer [Proc. London Math. Soc. (3) **8** (1958), 53–75; MR **19**, 1183], in which T is supposed to map the intersection of the unit ball with K into a compact set. In the case when K has non-empty interior, and T maps the non-zero points of K into the interior of K, Kreĭn and Rutman [ibid.] have obtained generalizations of other Frobenius theorems dealing with the finer structure of the spectrum of a positive matrix. Theorems of this type are considered here, especially those concerned with the simplicity of the eigenvalue 1. Let K be a normal cone with non-empty interior, and define $\bar\lambda$ as the sup of those λ for which there exists a positive vector x with $Tx\geqq\lambda x$, and similarly define $\underline{\lambda}$ in terms of the inequality $Tx\leqq\lambda x$. The relations between $\bar\lambda$, $\underline{\lambda}$ and the spectral radius of T are studied. This leads to some results on the perturbation of positive operators. Finally, there are some results concerning semi-groups of positive operators and their infinitesimal generators.

F. F. Bonsall (Newcastle-upon-Tyne)

4966:

Naĭmark, M. A. Decomposition of a tensor product of irreducible representations of the proper Lorentz group into irreducible representations. I. The case of a tensor product of representations of the fundamental series. Trudy Moskov. Mat. Obšč. **8** (1959), 121–153. (Russian)

The author gives the detailed proofs of results announced earlier [Dokl. Akad. Nauk SSSR **119** (1958), 872–875; MR **20** #7228]. In addition to describing the decomposition in question by specifying an appropriate measure and multiplicity function in the space of all irreducible representations he exhibits explicitly an integral operator which sets up the equivalence between the tensor product representation and the corresponding direct integral of irreducibles. In doing the latter he goes beyond the result of the reviewer alluded to in MR **20** #7228.

G. W. Mackey (Cambridge, Mass.)

4967:

Kasahara, Shouro. Representation of some topological algebras. III. Proc. Japan Acad. **35** (1959), 588–593.

The present paper is the third installment of the series of papers bearing the same title. The former two will be referred to by [I] and [II] [same Proc. **34** (1958), 355–360; **35** (1959), 89–94; MR **20** #6485; **21** #5908]. The main body of the paper is concerned with the following situation. Let E be a topological algebra satisfying the condition (ii) (see the review of [I]), and let p be an idempotent of rank 1 (see the review of [II]); then there is a pairing of linear spaces Ep and pE defined by $\langle xp, py\rangle p=p\,xy\,p$. A topological algebra E is said to be compatible with a non-zero idempotent of rank 1 if there is a non-zero idempotent p of rank 1 such that the dual of the vector space Ep (with the relative topology) is exactly pE under the pairing defined above. The main theorem of the paper states that if E is a locally convex Hausdorff algebra with (jointly) continuous ring multiplication, satisfying condition (ii), and compatible with a non-zero idempotent of rank 1, then there is a normed vector space X such that E is topologically isomorphic to a subalgebra (containing all transformations of finite rank) of $L_0(X, X)$, where $L_0(X, X)$ denotes the space $L(X, X)$ (see the review of [II]) with the topology of uniform convergence on bounded subsets of X. There are several corollaries to this theorem.

I. Namioka (Ithaca, N.Y.)

4968:

Sya, Do-Šin [Shah, Tao-shing]. On semi-normed rings with involution. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 509–528. (Russian)

This paper contains a complete exposition of results announced earlier for seminormed algebras R with involution [Dokl. Akad. Nauk SSSR **124** (1959), 1223–1225; MR **21** #5151]. For notation and terminology, see the review cited. In addition to the theorems stated in the review just cited, the following are proved.

Let R be a seminormed commutative algebra with unit e and involution *. Suppose that

$$\sup_{\alpha\in\mathfrak{A}}|x^*x|_\alpha=(\sup_{\alpha\in\mathfrak{A}}|x|_\alpha)(\sup_{\alpha\in\mathfrak{A}}|x|_\alpha)$$

for all $x\in R$, where $\mathfrak{A}$ denotes the collection of all seminorms $|x|_\alpha$ of R. Then R is algebraically and *-isomorphic with the algebra of all continuous complex-valued functions on the space $\mathfrak{M}$ of all maximal closed ideals in R, and each norm $|x|_\alpha$ corresponds to the maximum of the absolute value on some compact subset.

Let R be a complete seminormed algebra with involution and suppose that $\sup_{\alpha\in\mathfrak{A}}|x^*x|_\alpha=\sup_{\alpha\in\mathfrak{A}}|x|_\alpha^2$ for all $x\in R$. Then R is algebraically and *-isomorphic to an algebra of (not necessarily bounded) linear operators A_x on a certain Hilbert space H. The seminorms $|x|_\alpha$ are given by $\|P_\alpha A_x\|$, where P_α is a projection commuting with all A_x and $\|\ \|$ is the operator norm.

E. Hewitt (Seattle, Wash.)

4969:

Takeda, Zirô. On the extensions of finite factors. II. Proc. Japan Acad. **35** (1959), 215–220.

[For part I see Nakamura and Takeda, same Proc. **35** (1959), 149–154; MR **21** #7454.]

The author introduces cohomology theory for von Neumann factors of type II_1. Extensions of a factor are related to the group extensions of the inner automorphisms of the factor. The main theorem gives a one to one correspondence between extensions of a factor by a group of automorphism classes and a second cohomology group.

E. L. Griffin, Jr. (Ann Arbor, Mich.)

4970:

Zaidman, Samuel. La représentation des fonctions vectorielles par des intégrales de Laplace-Stieltjes. II. Tôhoku Math. J. (2) **12** (1960), 52–70.

[For part I see Ann. of Math. (2) **68** (1958), 260–277; MR **21** #803.]

The author publishes here the detailed proofs of results which he has announced earlier in C. R. Acad. Sci. Paris **247** (1958), 905–907; **248** (1959), 1915–1917 [MR **21** #1496, #1497]. *E. R. Lorch* (New York)

4971:

Schwartz, J. On Nash's implicit functional theorem. Comm. Pure Appl. Math. **13** (1960), 509–530.

The object of this paper is to extend known results concerning the question of when the range of a mapping $y=f(x)$ from a Banach space X to a Banach space Y, in which $y_0=f(x_0)$, contains a neighborhood of y_0 in Y. The author proves the following quite general theorem. Let M be a compact n-dimensional manifold of class C^∞. Let $f: C^k(M)\to C^0(M)$ be a non-linear mapping which is infinitely often differentiable in the Fréchet sense with $y_0=f(x_0)$. Let $g(x, \hat{x})$ be the Fréchet differential of f, and suppose that there is a neighborhood U of x_0 in $C^k(M)$ and a mapping $h(x, y): U\times C^k(M)\to C^0(M)$ which is linear in y such that $g[x, h(x, y)]=y$ for $x\in C^\infty(M)\cap U$ and $y\in C^\infty(M)$. Then $f[C^\infty(M)]$ covers a neighborhood of y_0 in $C^\infty(M)$.

The theorem is generalized by replacing the range space $C^0(M)$ by $L^1(M)$ and by requiring M and f to be differentiable only to some finite order, but the results are too detailed to state here.

C. B. Morrey, Jr. (Berkeley, Calif.)

CALCULUS OF VARIATIONS

See also 4869.

4972:

Reifenberg, E. R. Solution of the Plateau Problem for m-dimensional surfaces of varying topological type. Acta Math. **104** (1960), 1–92.

Let m and n be positive integers with $2\leq m<n$ and let Λ^m denote spherical Hausdorff measure on euclidean n-space. Roughly speaking the problem is to find among all sets S with given boundary one with smallest Λ^m-measure. In more precise terms, let G be a compact abelian group, A a compact set in n-space, and L a subgroup of the Čech homology group $H_{m-1}(A; G)$. Let $\mathcal{S}$ be the class of compact sets S such that $S\supset A$ and L is contained in the kernel of the inclusion homomorphism

$$i_*: H_{m-1}(A; G)\to H_{m-1}(S; G).$$

If A is a topological $(m-1)$-sphere, $G=$reals mod 1, and $L=H_{m-1}(A; G)$, then $\mathcal{S}$ is the class of sets $S\supset A$ such that A is not a retract of S. A set S is called proper if no proper subset of S belongs to $\mathcal{S}$. The main theorem is that $\Lambda^m(S)$ attains its minimum in $\mathcal{S}$, and if S is a proper minimizing set, then Λ^m-almost every point P of $S-A$ has a neighborhood homeomorphic with euclidean m-space. Moreover, such points P are characterized by the fact that S has m-density 1 at P; the remaining singular set consists of those points $P\in S-A$ where the m-density exceeds 1. For $n=2$, $m=3$, $G=$integers mod 2, L generated by a single element of $H_1(A; G)$, the singular set is shown to be empty, and $S-A$ is locally a minimal surface in the sense of classical differential geometry.

This version of the original Plateau problem was an open question of long standing, and was not amenable to the methods developed by Douglas, Rado, and others to solve the Plateau problem for surfaces defined by mappings from a 2-manifold of given finite topological type [see Courant's book, *Dirichlet's principle, conformal mapping, and minimal surfaces*, Interscience, New York, 1950; MR **12**, 90; and references cited there]. For $m>2$ the situation is now reversed. For example, scarcely anything is known about the problem of minimizing m-area among all surfaces given by mappings from an m-cube into n-space and bounded by a given topological $(m-1)$-sphere.

The following outline will give some idea of the proof of the main theorem. First of all, certain basic topological facts are needed to show that $\mathcal{S}$ is closed under such operations as taking sequential limits and cutting and patching holes. These facts are given in an Appendix by J. F. Adams. The next main step is an isoperimetric inequality for m-dimensional sets in n-space. A special minimizing sequence S_n is then constructed such that, if $S(P, r)$ denotes the solid n-sphere with center P and radius r, then

$$\Lambda^m[S_n\cdot S(P, r)]\geq br^m, \quad \text{for } r>r_n,$$

where b is a positive constant and $r_n\to 0$. A subsequence of S_n tends in the sense of Hausdorff distance to a limit S. By an argument involving cone constructions and a refined version of the Eilenberg inequality, Λ^m is lower semi-continuous on the special sequences S_n, whence S minimizes. We can take S to be proper. Then the set of points P where the m-density is one is open relative to S, and given any such P and $\varepsilon>0$ there exists $r_1>0$ such that if $X\in S$, $|XP|<r_1$ and $0<r<r_1$ then there is an m-plane Σ through X (depending on both X and r) such that the Hausdorff distance between $\Sigma\cdot S(X, r)$ and $S\cdot S(X, r)$ is $<\varepsilon r$. By an ingenious construction a sequence of topological m-discs S^j is then found such that, as $j\to\infty$, S^j tends to a topological m-disc containing P as an interior point. *W. H. Fleming* (Providence, R.I.)

GEOMETRY

4973:

Mandan, Sahib Ram. Orthogonal tetrahedron. Math. Mag. **31** (1957/58), 127–131.

The author considers the question whether a tetrahedron is orthocentric if its three bialtitudes (i.e., the common perpendiculars of its three pairs of opposite edges) are concurrent. No references regarding the question are mentioned. The problem was considered before more than once, and the solution arrived at by the author is at variance with the results obtained by the earlier authors [C. W. Merrifield, Question 3088, Educational Times, Reprints (1) **14** (1871), 25; A. Marmion, Mathesis **57** (1948), 8–17, 133–135; specifically p. 15, art. 6; bibliography, pp. 8–9].

N. A. Court (Norman, Okla.)

4974:

Caravelli, Vito. ★Le traité des hosoèdres. Traduit sur le texte latin original avec des notes par Paul Ver Eecke.

Librairie Scientifique et Technique, Albert Blanchard, Paris, 1959. 24 pp. 3.50 NF.

A hosohedron is the common part of the solid content of n congruent elliptic cylinders whose cross sections have a common minor (or major) axis b while the infinite axes of the cylinders join pairs of opposite vertices of a regular $2n$-gon in a plane perpendicular to b. Thus the hosohedron is a 'polyhedron' of type $\{2, 2n\}$ having 2 vertices, $2n$ 'edges' (semiellipses) and $2n$ 'faces' (each a part of the surface of a cylinder). Pairs of opposite 'edges' form ellipses all lying on a circumscribed spheroid, or possibly circles all lying on a circumscribed sphere (in which case we have an 'internal' hosohedron). The 'medians' of pairs of opposite 'faces' form ellipses all lying on an inscribed spheroid, or possibly circles all lying on an inscribed sphere (in which case we have an 'external' hosohedron). The author computes the volume of such a solid by the kind of procedure that was used by Archimedes, that is, without formal application of integral calculus.

H. S. M. Coxeter (Toronto)

4975:

Leminger, Otakar. On the regular 257-gon. Časopis Pěst. Mat. **84** (1959), 371–373. (Czech)

An algebraic presentation of the construction of the regular 257-gon with ruler and compass.

4976:

Glavas, Christos B. A contribution to the use of the polar system. Prakt. Akad. Athēnōn **33** (1958), 342–353 (1959). (Greek summary)

Remarks on interpretation of the sign of r.

4977:

Stamate, I. A conic related to a given conic. Inst. Politehn. Cluj. Lucrări Ştiinţ. **1** (1958), 79–81. (Romanian. Russian and French summaries)

Let

$$f(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0$$

be a given conic. Set $f_x = a_{11}x + a_{12}y + a_{13}$, $f_y = a_{12}x + a_{22}y + a_{23}$. The author studies how the type of conic given by $f(f_x, f_y) = 0$ is related to the type of conic $f(x, y) = 0$ originally given.

R. G. Stanton (Waterloo, Ont.)

4978:

Cunningham, F., Jr.; Valentine, Charles W. Axioms for the affine line. Amer. Math. Monthly **67** (1960), 549–559.

E. Artin, in his *Geometric algebra* [Interscience, New York, 1957; MR **18**, 553; p. 51] defined dilatations as "one-to-one maps of the plane onto itself which move all points of a line into points of a parallel line". For a purely one-dimensional theory, the question arises: What property can take over the role of parallelism? The authors find that one-dimensional dilatations can be defined as one-to-one ordered correspondences that form a doubly transitive group. They make much use of dilatations of period 2, which they call "inversions" (after the "central inversion" of the crystallographers).

H. S. M. Coxeter (Toronto)

4979:

Jůza, Miloslav. Familles développables d'homologies et de pseudohomologies. Czechoslovak Math. J. **9** (84) (1959), 579–589. (Russian summary)

S_n denotes the (real?) projective n-space, $n \geq 2$. A homography H is called a pseudohomology (and a homology if $k=0$) if it leaves pointwise fixed two complementary subspaces S_k, S_{n-k-1}, $0 \leq k < n-1$, which are called axes. Therefore, for a suitable base $A_0, \cdots, A_n$ we have $HA_i = \rho A_i$, $i = 0, \cdots, k$, and $HA_i = A_i$, $i = k+1, \cdots, n$, where ρ is the "characteristic". A family $\{H(t)\}$ of pseudohomologies, where t is a real variable, "has an envelope" if for each t there exist a real λ and a hyperplane ξ such that $H'X = \lambda HX$ for each X in ξ. Theorem: $\{H(t)\}$ has an envelope if and only if: (i) One of the axes is fixed (say S_k); (ii) the $(n-k)$-dimensional variety $\{S_{n-k-1}(t)\}$ traced out by the other axis is developable; (iii) if $k=0$, then ρ is constant. The author also examines the family $\{\xi(t)\}$.

E. C. Zeeman (Cambridge, England)

4980:

Segre, B. On complete caps and ovaloids in three-dimensional Galois spaces of characteristic two. Acta Arith. **5**, 315–332 (1959).

In einem projektiven r-dimensionalen Raum $S_{r,q}$ über einem Galoisfeld der Ordnung $q = p^h$ untersucht Verf. in Weiterführung früherer Arbeiten [insb. Canad. J. Math. **17** (1955), 414–416; Ann. Mat. Pura Appl. (4) **39** (1955), 357–379; Rev. Math. Pures Appl. **2** (1957), 289–300; MR **17**, 72, 776; **20** #4808] Mengen $k_{r,q}$ von k verschiedenen Punkten, von denen keine drei kollinear sind, die sog. k-Bögen ($r=2$), resp. k-Kappen ($r \geq 3$); sie heissen bei maximalem k zu gegebenem r, q-Ovale ($r=2$), resp. Ovaloide ($r \geq 3$). Ihre Struktur ist für ungerades q leicht zu übersehen im Gegensatz zum Falle $p=2$, der hier untersucht wird. Verf. gibt vorab ein Beispiel an für ein $k_{r,q}$ mit $q=4$, $k=12$ und mit $q=2^h$, $k=3q+2$, die beide jeweils nicht Untermenge eines $(k+1)_{r,q}$ sind.—Das Hauptergebnis der Arbeit ist, dass für $q = 2^h \geq 8$ Ovale des $S_{r,q}$ existieren, die keine Quadriken sind. Für $q=8$ wird auf analytischem Wege ein solches Beispiel explizit angegeben. Zur Herleitung dieses Resultates studiert Verf. die Polaritäten bezüglich eines Ovaloides des $S_{r,q}$ mit $q = 2^h \geq 4$ und die ebenen Schnitte eines Ovaloides, insbesondere die darin liegenden punktierten Kegelschnitte.

R. Moufang (Frankfurt)

4981:

Roth, Leonard. Alcune estensioni di un teorema di Enriques. Rend. Sem. Mat. Fis. Milano **26** (1954–55), 122–135.

L'A. espone il contenuto di una sua conferenza sul problema di determinare criteri di unirazionalità per le varietà algebriche. In particolare L'A. estende un classico teorema di Enriques [Ann. Mat. (3) **20** (1913), 109–111] che assicura l'unirazionalità di una varietà algebrica V_p, a p dimensioni, quando questa possegga una congruenza birazionale di curve razionali dotata di una varietà V_{p-1} n-secante birazionale, che sia semplice per V_p. L'estensione viene fatta sostituendo alla congruenza di curve una opportuna congruenza di V_k ($k \geq 2$), e data la sua ampia generalità, permette di fare molte interessanti applicazioni, che sono illustrate da numerosi esempi.

M. Rosati (Rome)

4982:

Salzano, Filomena. La varietà determinata da una quaterna ordinata da ipersuperfici dell' S_4 complesso, nell' S_{19}. Giorn. Mat. Battaglini (5) **5** (85) (1957), 244–263.

4983:

Vaona, Guido. Le trasformazioni fra piani che posseggono infinite coppie di curve simili. Boll. Un. Mat. Ital. (3) **15** (1960), 170–174. (English summary)

Author's summary: "Plane transformations which have an infinity of similar curves are determined."

4984:

Bompiani, Enrico. Sui sistemi omaloidici di quadriche nello spazio ordinario. Ann. Mat. Pura Appl. (4) **50** (1960), 285–289.

Author's summary: "I sistemi omaloidici di quadriche nell'ordinario spazio proiettivo sono stati determinati da L. Cremona quasi un secolo fa [Rend. Inst. Lombardo (2) **4** (1871), 269–279, 315–324; *Opere complete, Vol. 3*, pp. 241–259, U. Hoepli, Milan, 1917]; e il suo procedimento si trova riportato nei trattati sull'argomento [H. Hudson, *Cremona transformations in plane and space*, pp. 167–169, Univ. Press, Cambridge, 1927; F. Conforto, *Le superficie razionali*, pp. 17–22, Zanichelli, Bologna, 1939; L. Godeaux, *Les transformations birationnelles de l'espace*, no. 19, Gauthier-Villars, Paris, 1934].

"La stessa determinazione può raggiungersi con mezzi molto più elementari (che forse potranno esser utili in altri casi); è quello che mi propongo di mostrare qui appresso."

4985:

Predonzan, Arno. Alcune questioni di separabilità. Rend. Sem. Mat. Univ. Padova **30** (1960), 124–148.

Let F be an absolutely irreducible surface in three-dimensional projective space, defined over a subfield k of an algebraically closed field K of characteristic zero. Suppose that F has order greater than 2, and is not developable. Then at each simple point of F there are two inflexional tangents. F is said to be asymptotically separable if, in some extension of k, the set of pairs of inflexional tangents can be regarded as the set of pairs of an irreducible algebraic correspondence between two subsystems of inflexional tangents.

Necessary and sufficient conditions for this to happen are given. We quote the two following theorems.

(1) A necessary and sufficient condition for F to be asymptotically separable is that, if $h(X)$ is the hessian of F and x a generic point of F, $h(x)$ be the square of an element in $k(x)$ or in some quadratic extension of $k(x)$.

(2) Necessary and sufficient conditions for F to be asymptotically separable are (a) that each component of the parabolic curve Γ have even multiplicity as a component of the intersection of F and its hessian h; (b) that the curve $\frac{1}{2}\Gamma$, defined over k or a quadratic extension of k, be linearly equivalent to the curve cut on F by a surface whose order is one-half that of h. Condition (a) alone is not sufficient, the four-nodal cubic surface being a counter-example.

A number of particular examples are constructed and studied. *J. A. Todd* (Cambridge, England)

4986:

Predonzan, Arno. Una nuova caratterizzazione delle rigate cubiche ed alcuni problemi di classificazione. Rend. Sem. Mat. Univ. Padova **30** (1960), 161–177.

This paper is based on the preceding one [see preceding review], and concerns cubic surfaces in ordinary space. Theorems:

(I) The components of the parabolic curve of a non-ruled cubic surface F all have even multiplicity in the intersection of the surface and its hessian in the following cases, and in no others: (1) F possesses four conical nodes; (2) F possesses two conical nodes and a binode at which the line common to the two tangent planes lies on F; (3) F possesses a conical node and a binode, such that the line of intersection of the two tangent planes lies on F, and the common tangent plane to F at all points of this line has, for its intersection with F, the line itself counted with multiplicity 3; (4) F possesses three binodes.

(II) A non-ruled cubic surface is asymptotically separable [see previous review] if and only if it has three binodes. *J. A. Todd* (Cambridge, England)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 4935, 4996.

4987:

McMinn, Trevor J. On the line segments of a convex surface in E_3. Pacific J. Math. **10** (1960), 943–946.

Let C be a bounded convex solid in E^{n+1} with nonempty interior and let D be the set of all straight-segment subsets of the boundary of C, considered in the natural way as a subset of the sphere S^n. Answering some questions of V. L. Klee, the author proves that, for $n=2$, D is of first category and n-dimensional measure 0 in S^n.

Z. A. Melzak (Vancouver, B.C.)

4988:

Szenthe, J. Die Verallgemeinerung eines Satzes von H. Tietze. Acta Math. Acad. Sci. Hungar. **10** (1959), 397–404. (Russian summary, unbound insert)

Defining a euclidean point set P to be locally convex provided $p \in P$ implies the existence of a subset $K(p)$ of P that contains p, where $K(p)$ is convex (i.e., if r, $s \in K(p)$, $r \neq s$, $\mathrm{seg}(r, s) \subset K(p)$) and, for some positive number ρ, contains all points x of P with distance $px < \rho$, Tietze [Math. Z. **28** (1928), 697–707] proved that each closed, connected, and locally convex euclidean set P is convex. The purpose of this paper is to extend this theorem to subsets of a much wider class of spaces, replacing the classical notion of convexity of euclidean sets by the more general one of metric convexity (i.e., a subset S of a metric space M is metrically convex provided $p, r \in S$, $p \neq r$ imply the existence of $q \in S$, $p \neq q \neq r$ and $pq + qr = pr$). A subset S is locally convex provided for each of its points p a $\rho(p) > 0$ exists such that $B(p; \rho(p)) \cap S$ is (metrically) convex, where $B(p: \rho(p))$ denotes the set of all points x of M with distance $px \leqq \rho(p)$. A metric space is called straight provided every locally isometric mapping of a line segment onto a subset of the space is an isometric mapping of the segment onto the subset (i.e., all locally segmental subsets are metric

segments). The desired extension of Tietze's theorem establishes that closed, connected, locally convex subsets of a finitely compact metric space are convex if and only if the space is straight. {Reviewer's note. The author seems unaware of the following theorem, stated without proof in 1940 by C. Pauc [Rev. Sci. **78** (1940), 233; MR **2**, 260]: If M is a complete connected metric space that admits for each of its points a closed metrically convex neighborhood, each two distinct points p, q of M are joined by an arc that is locally a metric segment. If, moreover, M has the two triple property (that is, if two triples of a point-quadruple are linear, then so are the two remaining triples) then M is convex. This extension of Tietze's theorem is very closely related to the extension proved in this note.} *L. M. Blumenthal* (Columbia, Mo.)

4989:

Grünbaum, B. Some applications of expansion constants. Pacific J. Math. **10** (1960), 193–201.

Let X be a metric space. For $x \in X$ and $\rho \geqq 0$, let $S(x; \rho)$ denote the cell $\{y \in X : d(x, y) \leq \rho\}$. The expansion constant $E(X)$ of X is the infimum of numbers μ with the property that $\bigcap_{\alpha \in A} S(x_\alpha; \mu\rho_\alpha) \neq \emptyset$ for any family $S(x_\alpha; \rho_\alpha)$, $\alpha \in A$, of pairwise intersecting cells in X. For a retraction f of a metric space Y onto a subspace X, the norm $\|f\|$ of f is the infimum of numbers μ such that $d(f(y), f(y')) \leq \mu d(y, y')$ for all $y, y' \in Y$. The retraction constant $r(X)$ of a metric space X is the infimum of numbers μ with the following property: for every metric space Y which contains X and only one point not in X, there is a retraction f of Y onto X with $\|f\| \leq \mu$. For a normed linear space X, the projection constant $p(X)$ of X is the infimum of numbers μ with the following property: for any normed linear space Y containing X as a subspace of deficiency 1, there is a projection P of Y onto X with $\|P\| \leq \mu$. If the constant $E(X)$ (or $r(X)$ or $p(X)$) defined as an infimum is actually a minimum, then it is said to be exact. The following results are proved. (I) For any normed linear space X, we have $E(X) = p(X)$. If one of them is exact, so is the other. (II) $E(X) \leq r(X)$ holds for any metric space X. If a metric space X is metrically convex in the sense of K. Menger [Math. Ann. **100** (1928), 75–163], then $E(X) = r(X)$ and they are simultaneously exact or not exact. These results are applied to derive a theorem of L. Nachbin [Trans. Amer. Math. Soc. **68** (1950), 28–46; MR **11**, 369] on the equivalence of the Hahn-Banach extension property of a Banach space and the binary intersection property, and a theorem of N. Aronszajn and P. Panitchpakdi [Pacific J. Math. **6** (1956), 405–439; MR **18**, 917] on an extension property for uniformly continuous mappings. *Ky Fan* (Detroit, Mich.)

DIFFERENTIAL GEOMETRY

See also 5050, 5051.

4990:

Pogorelov, A. V. ★Differential geometry. Translated from the first Russian ed. by L. F. Boron. P. Noordhoff N. V., Groningen, 1959. ix+171 pp. Paperbound: $3.90.

The original is *Lekcii po differencial'noĭ geometrii* [Izdat. Har'kov. Gosudarstv. Univ., Kharkov, 1955]. A textbook on the elements of differential geometry (in Euclidean 3-space), up through the standard facts on first and second fundamental forms of a surface, and intrinsic geometry, including the Gauss-Bonnet theorem (for a region homeomorphic to a disk). "The author's aim is to present a rigorous discussion . . . without disturbing well-established tradition. A large amount of factual material . . . has been relegated to exercises and problems."

4991:

Mihăileanu, N. Relations entre les formules des géométries différentielles euclidienne et non-euclidiennes. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **10** (1959), 327–335. (Romanian. Russian and French summaries)

Let x_0, x_1, x_2, x_3 be homogeneous coordinates of a generic point M and $q^2x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$ the equation of the absolute in a noneuclidean space. Then one passes formally from a relationship in noneuclidean differential geometry to one in euclidean by (a) placing $x_0 = 1$, (b) multiplying the coordinates of any point conjugate to M by q, and (c) letting $q \to \infty$. The author also shows how to pass formally from a relationship written in point form in euclidean differential geometry to one in noneuclidean. The essential step is to replace the second derivatives that appear in a specified way. *A. Schwartz* (New York)

4992:

Mihăileanu, N. N. Un cas de dualité dans la théorie des courbes et des surfaces. Com. Acad. R. P. Romîne **9** (1959), 679–685. (Romanian. Russian and French summaries)

Let M be a point on a curve in a noneuclidean space of curvature $1/q^2$ and let T, N, B be the points conjugate to M on the tangent, principal normal, and binormal. Let ds be the element of arc, $1/\rho$ the curvature, and $1/\tau$ the torsion of the curve. Then from each metric relation one gets another by the substitutions $M \leftrightarrow B$, $T \leftrightarrow N$, $ds \to -(q/\tau)ds$, $\rho \to q(\rho/\tau)$, $\tau \to q^2/\tau$. To illustrate, the Frenet equations $dM/ds = T/q$, $dB/ds = -N/\tau$ and $dT/ds = -M/q + N/\rho$, $dN/ds = -T/\rho + B/\tau$ are respectively dual.

Let M be a point on a surface and M' the pole of the tangent plane at M with respect to the absolute quadric. Let e, f, g; l, m, n; e', f', g' be the coefficients of the three fundamental forms. From each metric relation one gets another by the substitutions $M \leftrightarrow M'$; $e, f, g \leftrightarrow e', f', g'$; $l, m, n \leftrightarrow l, m, n$. The author illustrates with the equations of Gauss and Weingarten and the Codazzi equations.

A. Schwartz (New York)

4993:

Drăgilă, Pavel. Correspondances singulières par parallélisme des plans tangents des deux surfaces. Proc. Amer. Math. Soc. **9** (1958), 518–521.

This paper is concerned with couples of surfaces whose tangent planes are parallel at corresponding points. The author studies the problem of finding if there exist couples of surfaces between whose points such a parallelism has been established and such that there exists on one surface a direction, tangent to a coordinate line, which shall be parallel to the corresponding direction on the other corresponding surface. He actually determines surfaces for which this holds.

E. T. Davies (Southampton)

4994:

Drăgilă, Pavel. Transformations singulières des surfaces. Bull. Sci. Math. (2) **82** (1958), 41–48.

This paper is concerned with the correspondence between two surfaces such that the tangent planes are parallel at corresponding points. Peterson [Ann. Fac. Sci. Univ. Toulouse (2) **7** (1905), 5–263] had stated that in such a correspondence there is always a couple of directions on one surface which are parallel to the corresponding directions on the other surface. The present author [#4993] has proved that such a couple of directions does not exist. In the present paper he gives a more general proof of the fact that upon two surfaces on which tangent planes at corresponding points are parallel only one system of parallel tangents exist. He proves further theorems in the same direction, in particular that the system of parallel tangents determines a system of asymptotic curves. *E. T. Davies* (Southampton)

4995:

Calapso, Maria Teresa. Sulle congruenze paraboliche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **26** (1959), 757–762.

This note continues the study of problems studied by P. Drăgilă [#4993] and B. Segre [same Atti **25** (1958), 381–388; MR **22** #2939]. The author considers a surface parametrised by u and v immersed in an affine space of n dimensions. Let the curves $v=$constant constitute a family of asymptotic lines. The tangents to these asymptotic lines generate a parabolic congruence of which the surface considered is the only focal surface.

A family of asymptotic lines and a parabolic congruence are said to be conjugate if the ray which describes the congruence always passes through the point of the surface which generates the family, and if, further, to the developables of the congruence, there correspond the curves of the family of asymptotic lines. Two families of asymptotic lines are said to be parallel if there is a correspondence between the two supporting surfaces such that the tangents to the relative asymptotic lines at corresponding points are parallel. In connection with this last notion, Segre has proved that if two families of asymptotic lines are parallel, the line joining the corresponding points describes a congruence conjugate to both families.

The main theorem proved in this note is as follows. In order to determine, in the most general manner, a parabolic congruence which is conjugate to a given family of asymptotic lines, it is sufficient to determine any family parallel to the given one; then the straight line joining the point which describes the given family to the corresponding point of the parallel family always generates a conjugate congruence. *E. T. Davies* (Southampton)

4996:

Klingenberg, Wilhelm. Neue Ergebnisse über konvexe Flächen. Comment. Math. Helv. **34** (1960), 17–36.

Let M be a closed surface with positive Gaussian curvature K, and let $L(G)$ be the length of a closed geodesic segment on M. Then it is proved that $2\pi/\sqrt{\max K} \leq L(G)$ where equality holds only on a sphere. This theorem complements the theorem of Toponogow to the effect that $L(G) \leq 2\pi/\sqrt{\min K}$.

For a point p of M, let $C(p)$ be the locus of minimum points and $d(p, C(p))$ the distance from p to this locus. Further let $d(M)$ be the maximum distance between any pair of points on M measured on M. Then it is proved that $d(p, C(p)) \geq \pi/\sqrt{\max K}$ and $d(M) \geq \pi/\sqrt{\max K}$. Again equality holds only on a sphere. The parallel theorem of Toponogow is that $d(M) \leq \pi/\sqrt{\min K}$.

The outer diameter $D(M)$ is the maximum distance between any pair of points on M measured in the enveloping E^3. Then it is proved that

$$2/\sqrt{\max K} \leq D(M) < \pi/\sqrt{\min K},$$

where the equality on the left holds only for a sphere.

C. B. Allendoerfer (Seattle, Wash.)

4997:

Chern, Shiing-Shen. Integral formulas for hypersurfaces in Euclidean space and their applications to uniqueness theorems. J. Math. Mech. 8 (1959), 947–955.

By using an inequality of L. Gårding [#4809 above] and deriving some new integral formulas, the author obtains a general translation theorem on closed strictly convex hypersurfaces in an $(n+1)$-dimensional Euclidean space E_{n+1}, from which the following two theorems are deduced. Theorem 1 (Alexandroff-Fenchel-Jessen): Two closed strictly convex hypersurfaces M_n and M_n^* in E_{n+1} are congruent under a translation, if the lth $(1 \leq l \leq n)$ elementary symmetric functions P_l and P_l^* of the radii of principal curvatures of M_n and M_n^*, respectively, take the same value at each pair of points of M_n and M_n^* with the same unit normal vector. Theorem 2: A closed strictly convex hypersurface in E_{n+1} with $P_{l-1}^{\alpha} P_l^{\beta} =$ const., $2 \leq l \leq n$, $\alpha+\beta>0$, $\alpha \geq 0$, $\beta \geq 0$, is a hypersphere. An extension of these two theorems to hypersurfaces with boundary is also remarked.

C. C. Hsiung (Bethlehem, Pa.)

4998:

Gołąb, S. On the notion of gradient. I. Essentiality of regularity suppositions. Ann. Polon. Math. 8 (1960), 1–4.

If σ is a C^1 function of n variables, it is well known that $\partial\sigma/\partial x_i$ are components of a vector field, the gradient. If, however, σ is merely differentiable (without continuous derivatives) these expressions are not necessarily components of a vector. An example for $n=2$ is given in detail. *C. B. Allendoerfer* (Seattle, Wash.)

4999:

Gołąb, S.; Kucharzewski, M. On the notion of gradient. II. A certain extremal property of direction of the gradient vector. Ann. Polon. Math. 8 (1960), 5–12.

An interpretation of the gradient vector is given as follows. Assume that σ is continuous near a point p, that σ has a total differential at p, that the vector $v_i = (\partial\sigma/\partial x_i)_p$ is non-zero, and that there exists a sequence of hypersurfaces S_ν with the properties: (I) S_ν is a closed $(n-1)$-dimensional surface containing p internally; (II) if β_ν is the maximum distance of S_ν from p, then $\lim_{\nu\to\infty} \beta_\nu = 0$; (III) if α_ν is the minimum distance of S_ν from p, then $\lim_{\nu\to\infty} \beta_\nu/\alpha_\nu = 1$. Let μ_ν be the maximum value of σ on S_ν, y_ν a point on S_ν such that $\sigma(y_\nu) = \mu_\nu$, and r_ν the radius from p to y_ν. Then it is proved that the sequence r_ν is convergent and that the limiting radius passes through the vector v.

This interpretation may be used to define a gradient

direction when the gradient does not exist in the classical sense. Examples are given.

C. B. Allendoerfer (Seattle, Wash.)

5000:

Goląb, S.; Pliś, A. On the notion of gradient. III. Gradient as a limit value of a surface integral. Ann. Polon. Math. **8** (1960), 13–22.

From the formula $\int_\Delta \operatorname{grad}\sigma = \int_S N\cdot\sigma$ it is customary to derive the statement that $\operatorname{grad}\sigma(p) = \lim_{S\to p} V^{-1}\int_S N\cdot\sigma$. In order for this derivation to be correct, assumptions about the limiting process must be made. The authors give a set of hypotheses which are sufficient and several counterexamples to show why they must be imposed.

C. B. Allendoerfer (Seattle, Wash.)

5001:

Weier, Joseph. Sur la divergence et la rotation de champs tensoriels. C. R. Acad. Sci. Paris **250** (1960), 1958–1959.

The author rediscovers several theorems of Bidal and de Rham [Comment. Math. Helv. **19** (1946), 1–49; MR 8, 93]. He then states three theorems on closed and coclosed forms which are related to the theorem of Stokes.

C. B. Allendoerfer (Seattle, Wash.)

5002:

Goląb, S.; Kucharzewski, M. Ein Beitrag zur Komitantentheorie. Acta Math. Acad. Sci. Hungar. **11** (1960), 173–174. (Russian summary, unbound insert)

The authors prove the theorem: If $T^{\lambda_1\cdots\lambda_r}{}_{\mu_1\mu_2\ldots\mu_s}$ is a (p, q) tensor, $p+q$ odd, which is an algebraic comitant of the tensor g_{ik}, then T::: is zero. The same holds for such a tensor if it is an algebraic comitant of the tensor $g^{\lambda_1\lambda_2}\cdots\lambda_r{}_{\mu_1\mu_2\ldots\mu_s}$, where $r+s$ is even.

E. M. Bruins (Amsterdam)

5003:

Fichera, Gaetano. Misure tensoriali e omologia su una varietà differenziabile. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **26** (1959), 466–477.

In a compact coordinate space V_r of class C^1 take a tensor $f_{i_1\ldots i_k}$, say continuous. In any fixed coordinate neighborhood $P:(x)$, one can introduce, for any Borel set $B\subset P$, the object

$$(1)\qquad \mu_{i_1\ldots i_k}(B) = \int_B f_{i_1\ldots i_k}(x)\,dx_1\cdots dx_k,$$

which for each $(i_1, \cdots, i_k)$ is an additive set function in $\{B\}$. Also, if two neighborhoods $P:(x)$, $Q:(y)$ intersect, $x=\varphi(y)$, and $B_1\subset P$, $B_2\subset Q$, $B_1=\varphi(B_2)$, then

$$(2)\qquad \mu_{j_1\ldots j_k}(B_2) = \int_{B_1}\left|\frac{\partial(y)}{\partial(x)}\right|\cdot\frac{\partial x^{i_1}}{\partial y^{j_1}}\cdots\frac{\partial x^{i_k}}{\partial y^{j_k}}\,d\mu_{i_1\ldots i_k}.$$

Now, the author introduces directly a tensorial additive set function $\mu_{i_1\ldots i_k}$ by the transformation law (2), calling it a k-measure. Also, he calls it regular if it admits a representation (1) by a point tensor f. A distinctive statement is the following. To each $(r-k)$-chain γ there corresponds one-one a k-measure t_γ, such that for any regular $(r-k)$-form ν,

$$\int_\gamma \nu = \int_{V_r} \nu\wedge t_\gamma.$$

There are many concepts and statements corresponding to those of the ordinary theory, and as a sample we mention the following. A $(k+1)$-measure α is m-homologous to 0 if and only if $\int_{V_r}\nu\wedge\alpha = 0$ for each $(r-k-1)$-form ν which is regular. Also, in such a case α is the external derivative of a k-form μ which is absolutely continuous.

S. Bochner (Princeton, N.J.)

5004:

Grabiel, Federico. Set functions and tensors. Tensor (N.S.) **10** (1960), 1–20.

At each point of space a tensor has a component in each coordinate system. Each component of a "set tensor" however refers to a set in space. When the set reduces to a point, one returns to the ordinary tensor theory.

Let C be the class of all linearly ordered systems whose elements belong to a field F. In § 2, the author defines "ordered sum" and "ordered product" in the cartesian product space $C\times F$ and then the integral of a point function over a set S. In § 3, a point function is "normalized" over a set S by taking a sort of average value over S. In § 4, set tensors are introduced. A contravariant set tensor of the first rank has components which transform according to the law $\overline{T}^i = \sum_k N(\partial\overline{x}^i/\partial x^k)\odot T^k$, $N(\partial\overline{x}^i/\partial x^k)$ being the normalization over S of the point function $\partial\overline{x}^i/\partial x^k$ and $\odot$ indicating "ordered product". The properties of ordinary point tensors are carried over to set tensors except for the transitive law. As a result, one must deal with "privileged" reference systems in this theory. In § 5 the algebra of set tensors is studied. In § 6 a displacement derivative which generalizes the partial derivative is introduced, and in § 7 a covariant derivative of a set tensor which is in turn a set tensor.

The expressions of this calculus of set functions and tensors follow closely those of ordinary theory, the coefficients of the law of transformation being normalizations of the ordinary coefficients, as in the equation quoted above.

A. Schwartz (New York)

5005a:

Stoka, Marius I. Géométrie intégrale dans un espace E_n. Rev. Math. Pures Appl. **4** (1959), 123–156.

5005b:

Stoka, Marius I. Famiglie di varietà misurabili in uno spazio E_n. Rend. Circ. Mat. Palermo (2) **8** (1959), 192–205.

5005c:

Stoka, Marius I. Congruences de variétés mesurables dans un espace E_n. Rev. Math. Pures Appl. **4** (1959), 431–449.

A transformation group G which acts in the euclidean space E_n is said to be "measurable" if there exists in E_n an integral invariant with respect to G which is unique up to a constant factor. The author obtains, in paper (a), all measurable groups for $n=1, 2, 3$. The method consists in taking all groups of 1, 2 or 3 variables (given by Sophus Lie) and testing those which admit an integral invariant.

Let F_q be a family of p-dimensional varieties of E_n depending upon q parameters. Let G_r be an r-dimensional

transformation group which acts in E_n and leaves the whole family F_q invariant. To this group corresponds in a natural way an isomorphic transformation group H_r which acts in the q-dimensional space of parameters of the family F_q. The problem of the author is to find all families of varieties F_q and groups G_r whose corresponding groups H_r are measurable. He gives, in paper (b), some general theorems and solves the problem explicitly for the following cases: $q=1$, $p=n-1$; $n=2$, $q=2$, $p=1$. In paper (c) the above problem is solved explicitly for the following cases: $n=2$, $q=1$, $p=1$; $n=3$, $q=1$, $p=2$; $n=3$, $q=2$, $p=1$. *L. A. Santaló* (Buenos Aires)

5006:

Stoka, Marius I. Sur les groupes de mouvement des espaces riemanniens V_2 à courbure constante. Acad. R. P. Romîne. Stud. Cerc. Mat. **11** (1960), 207–228. (Romanian. Russian and French summaries)

The author compares different forms (obtained one from the other by suitable transformations of coordinates) of the infinitesimal transformations of the groups of motions of a two-dimensional riemannian variety of constant curvature, and verifies that these forms and some properties of the groups of motions can be deduced from the structural equations of the groups.

L. A. Santaló (Buenos Aires)

5007:

Freudenthal, Hans. Zu den Weyl-Cartanschen Raumproblemen. Arch. Math. **11** (1960), 107–115.

The Raumproblem has to do with the characterization of a Riemann metric. Laugwitz [Arch. Math. **9** (1958), 128–133; MR **21** #2267] has stated the first and second formulations of this problem as given by H. Weyl, and has asserted that they are not equivalent. The present author disagrees and uses the method of E. Cartan to prove that the solution of the first problem is closely related to that of the second.

C. B. Allendoerfer (Seattle, Wash.)

5008:

Nagano, Tadashi; Takahashi, Tsunero. Homogeneous hypersurfaces in euclidean spaces. J. Math. Soc. Japan **12** (1960), 1–7.

Generalizing the reviewer's result [Trans. Amer. Math. Soc. **88** (1958), 137–143; MR **20** #2768] on compact homogeneous hypersurfaces, the authors prove that a homogeneous Riemannian manifold M of dimension n which can be isometrically immersed into the Euclidean space of dimension $n+1$ is the Riemannian direct product of a sphere and a Euclidean space except in the case when the second fundamental form is of rank 2 everywhere on M (i.e., the Gauss spherical map is of rank 2 everywhere on M). The main difficulty in the exceptional case which is unsettled is that an isometry of M may not come from a motion of the ambient Euclidean space.

S. Kobayashi (Vancouver, B.C.)

5009:

Yano, Kentaro; Takahashi, Tsunero. Some remarks on Einstein spaces and spaces of constant curvature. J. Math. Soc. Japan **12** (1960), 89–96.

Let M be a Riemannian manifold with metric $ds^2=\sum g_{\lambda\mu}dx^\lambda dx^\mu$ which is not necessarily positive definite. Consider the endomorphism of the space of p-forms defined by

$$\sum w_{\lambda_1\dots\lambda_p}dx^{\lambda_1}\wedge\cdots\wedge dx^{\lambda_p}\to(g^{\mu\nu}\nabla_\nu\nabla_\mu w_{\lambda_1\dots\lambda_p})dx^{\lambda_1}\wedge\cdots\wedge dx^{\lambda_p}.$$

Main results: (Theorem 3) If $\dim M\geq 3$, then the following conditions are equivalent: (a) M is an Einstein space; (b) the above endomorphism sends the space of closed 1-forms into itself; (c) the endomorphism sends the space of coclosed 1-forms into itself. (Theorem 4) If $\dim M\geq 3$ and $\neq 4$, then the following conditions are equivalent: (a) M is a space of constant curvature; (b) the endomorphism sends the space of closed 2-forms into itself; (c) the endomorphism sends the space of coclosed 2-forms into itself. All the results are obtained by local computations. Theorem 3 strengthens results of A. Avez [C. R. Acad. Sci. Paris **248** (1959), 1113–1115; MR **20** #7301] who proved the equivalence of (a) and (b) and that of (a) and (c) under the additional assumption that M is compact.

S. Kobayashi (Vancouver, B.C.)

5010:

Pinl, M. On affinely connected manifolds whose torsion can be transformed into constant components. Proc. Amer. Math. Soc. **11** (1960), 505–510.

Let A_n ($n\geq 3$) be an affinely connected space with torsion tensor $(t^\lambda{}_{\alpha\beta})$ such that the tensor $(t_{\alpha\beta})=(t^\lambda{}_{\alpha\mu}t^\mu{}_{\beta\lambda})$ is non-singular. It is proved that a coordinate-system exists with respect to which the components of the torsion tensor are constant functions if and only if the curvature tensor with metric tensor $(t_{\alpha\beta})$ is zero and the torsion is covariant constant with respect to this metric.

T. J. Willmore (Liverpool)

5011:

Tachibana, Shun-ichi. A remark on linear connections with some properties. Nat. Sci. Rep. Ochanomizu Univ. **10** (1959), 1–6.

If ϕ is a regular tensor field of type (1, 1) on a manifold, a linear connection is called a ϕ-connection if it makes ϕ covariantly constant. A ϕ-connection is constructed for any ϕ which satisfies $\phi^p=\varepsilon I$, $\varepsilon=\pm 1$, for some integer p. Existence theorems for connections related to almost-quarternionic structures and general almost-product structures, given by M. Obata [Japan J. Math. **26** (1956), 43–77; MR **20** #1796a] and the reviewer [Quart. J. Math. Oxford Ser. (2) **6** (1955), 301–308; MR **19**, 312], are deduced from a general construction, and the resulting connections are expressed in a more concise way.

A. G. Walker (Liverpool)

5012:

Lichnerowicz, André. Isométries et transformations analytiques d'une variété kählérienne compacte. Bull. Soc. Math. France **87** (1959), 427–437.

This is a sequel to the author's three previous notes [C. R. Acad. Sci. Paris **244** (1957), 3011–3012; **245** (1957), 953–956; **247** (1958), 855–857; MR **20** #996; **21** #4451, 4452]. Let V be a compact Kaehler manifold, L_a the Lie algebra of infinitesimal (complex) analytic transformations of V, L_1 the Lie algebra of infinitesimal isometries of V, and L_2 the complex subalgebra of L_a generated by L_1. Various subalgebras and ideals of L_a and L_2 and various decompositions of L_a are studied. A definite result is obtained when the scalar curvature of V is constant; namely, $L_a=L_2=C+I$, where C is the ideal of parallel

vector fields and I is the ideal of analytic vector fields X such that $h(X)=0$ for all holomorphic 1-forms h. This generalizes a theorem of Matsushima [Nagoya Math. J. **11** (1957), 145–150; MR **20** #995].

S. Kobayashi (Vancouver, B.C.)

5013:

Hiramatu, Hitosi. On n-dimensional Finslerian manifolds admitting homothetic transformation groups of dimension $>\frac{1}{2}n(n-1)+1$. Kumamoto J. Sci. Ser. A **4**, 4–10 (1959).

This paper is concerned with the Finslerian manifolds M which admit a group of homothetic transformations of "sufficiently high" dimensions. The following are proved. (1) If $\dim M=n\neq 4$, then M admits no effective group G of homothetic transformations which is not a group of motions such that $\frac{1}{2}n(n-1)+2<\dim G<\frac{1}{2}n(n+1)+1$. (2) Let G be an effective group of homothetic transformations of M. If $\dim G=\frac{1}{2}n(n+1)+1$, then M is locally euclidean. If $n\neq 4$, and $\dim G=\frac{1}{2}n(n-1)+2$, then M is Minkowskian.

H.-C. Wang (Evanston, Ill.)

GENERAL TOPOLOGY, POINT SET THEORY

5014:

Titus, C. J. A theory of normal curves and some applications. Pacific J. Math. **10** (1960), 1083–1096.

A complex-valued function $\zeta(t)$ on a real interval is a regular representation of a curve if $\zeta'(t)$ is continuous and non-vanishing; and such a representation is normal (Whitney) if it is one-to-one except for a finite number of simple crossing points. In this paper relations between the self-intersections of a curve are studied and applied to describe various properties of the curve and also to show that the normal curves are dense, in a certain sense, among the regular curves. Among other results the author obtains (i) a criterion that a curve be simple in terms of tangent winding numbers and circulation index, (ii) a condition under which $\zeta(t)$ represents a positively oriented Jordan curve and (iii) a condition under which a light open mapping on a disk will be a homeomorphism, stated in terms of the curve representation which it gives on the edge of the disk.

G. T. Whyburn (Charlottesville, Va.)

5015:

Fort, M. K., Jr. Mappings on S^1 into one-dimensional spaces. Illinois J. Math. **1** (1957), 505–508.

Sind X, X' topologische Räume und f eine eindeutige Abbildung von X in X', so heisse f lokal eineindeutig, wenn zu jedem Punkte p aus X eine Umgebung U von p existiert derart, dass $f|U$ eineindeutig ist. Dann beweist Verf. unter anderem: Wenn S eine Kreislinie und g eine lokal eineindeutige stetige Abbildung von S auf den eindimensionalen metrischen Raum Y ist, so ist g wesentlich.

J. Weier (Zbl **77**, 362)

5016:

Loibel, Gilberto Francisco. On two definitions of quotient space. Bol. Soc. Mat. São Paulo **11** (1956), 115–121 (1959). (Portuguese)

Let R be an equivalence relation on a space E. The author gives a necessary and sufficient condition on R in order that the neighborhoods for E/R, according to a definition he ascribes to Seifert-Threlfall, be neighborhoods (i.e., satisfy Bourbaki's axioms). Actually, the definition given is in terms of neighborhoods (of E) rather than what Seifert-Threlfall called neighborhoods, and is not equivalent with theirs.

L. Gillman (Rochester, N.Y.)

5017:

Hulanicki, A. On the power of compact spaces. Colloq. Math. **7** (1959/60), 199–200.

If every point in a compact space has character $\geq m$ (m infinite), then the cardinal of the space is at least 2^m. The author acknowledges an earlier proof by Mrówka [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 89–93; MR **20** #4821].

L. Gillman (Rochester, N.Y.)

5018:

Mrówka, S. On the potency of subsets of βN. Colloq. Math. **7** (1959), 23–25.

Most results contained in the note are known (as pointed out by the author). The following one, although essentially known, seems not to have been explicitly stated before: every infinite pseudocompact subset of βN is of power $\geq \mathfrak{c}$, and there exists a countably compact subset of βN of the power $\mathfrak{c}$.

M. Katětov (Prague)

5019:

Isiwata, Takesi. On locally Q-complete spaces. II, III. Proc. Japan Acad. **35** (1959), 263–267, 431–434.

The definitions of locally Q-complete space and one-point Q-completion are indicated in conditions (1) and (3) listed in an earlier review [MR **21** #5945] of same Proc. **35** (1959), 232–236. In II, it is shown that if X is a locally Q-complete space, then a subring γ of $C(X)$ can be found with the property that the kernel of any homomorphism of γ onto the real field is the ideal of all functions in γ that vanish at some point of X. From this it is deduced that the set of all such ideals in γ, provided with the hull-kernel topology, is homeomorphic with X. The main result in III is that if X is a locally Q-complete space but does not coincide with its completion υX in the uniform structure generated by $C(X)$, then there is a natural one-to-one correspondence between the family of all one-point Q-completions of X and the family of all compact subsets of $\beta X-X$ that contain $\upsilon X-X$. {In II, statement (iii) of § 4 and the corollary to theorem 3 are false, as shown by the sum of a countable number of copies of $W(\omega_1)$.}

C. W. Kohls (Rochester, N.Y.)

5020:

Isiwata, Takesi. On ring homomorphisms of a ring of continuous functions. Proc. Japan Acad. **35** (1959), 268–272.

Most of the statements in the paper fall into one of the following categories: (1) alternative proof of a known theorem [theorems 3 and 4]; (2) trivially true statement [theorem 1, corollaries 1, 3 and 5]; (3) false statement [§ 1, (3) and (6), theorem 2]. The remaining results are: $C(X)$ [$C^*(X)$] has a subalgebra γ that generates a uniform

structure on X, with the property that the kernel of any homomorphism of γ onto the real field is the ideal of all functions in γ that vanish at some point of X, if and only if X is a locally Q-complete [locally compact] space. [See #5019 above.] *C. W. Kohls* (Rochester, N.Y.)

5021:

Isiwata, Takesi. On ring homomorphisms of a ring of continuous functions. II. Proc. Japan Acad. **35** (1959), 325–339.

In a paper on characterizations of subrings of $C(X)$ [Illinois J. Math. **3** (1959), 121–133; MR **20** #7214], F. W. Anderson and R. L. Blair pointed out that in the statement of their first characterization theorem, the term "completely regular" cannot be replaced by "Q-space". It is shown here that when A is an algebra, the term can be replaced by "locally Q-complete space" [see #5019 above]. A result similar to the second characterization theorem of their paper is also obtained, with a positive function bounded away from zero replacing the identity. Finally, characterizations of various kinds of spaces X are given in terms of the collection of subalgebras [resp. ideals] in $C(X)$ that are weakly pseudo-regular and point-determining in the sense of Anderson and Blair. Statement (4) in theorem 4 is false, however—let X be the real numbers and consider the set of fixed maximal ideals in $C(X)$. *C. W. Kohls* (Rochester, N.Y.)

5022:

Vilenkin, N. Ya. On a theorem of F. Hausdorff. Moskov. Gos. Ped. Inst. Uč. Zap. **108** (1957), 73–74. (Russian)

Author's introduction: "A simple proof that for a metrizable space to be compact it is necessary and sufficient that it be bounded in an arbitrary metric."

5023:

Bingen, Franz. Métrisation des espaces topologiques. Bull. Soc. Math. Belg. **9** (1957), 107–114.

Expository paper proving the sufficiency of the metrization criterion of Bing [Canad. J. Math. **3** (1951), 175–186; MR **13**, 264], Nagata [J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **1** (1950), 93–100; MR **13**, 264], and Smirnov [Dokl. Akad. Nauk SSSR **77** (1951), 197–200; MR **12**, 845]. The paper begins with some historical remarks {additional references: A. H. Frink, Bull. Amer. Math. Soc. **43** (1937), 133–142; and J. Nagata, J. Inst. Polytech. Osaka City Univ. Ser. A **8** (1957), 185–192; MR **20** #4256}, and ends with an analogue of Uyrsohn's imbedding theorem [cf. C. H. Dowker, Duke Math. J. **14** (1947), 639–645; MR **9**, 196]. {On p. 108, line 7*, the second W should be W^{-1}. At the foot of p. 109, the functions f_t should be assumed continuous.} *A. H. Stone* (Manchester)

5024:

Aleksandrov, P. On the metrisation of topological spaces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **8** (1960), 135–140. (Russian. English summary)

Author's summary: "The author gives new conditions in which a topological space is metrisable. He introduces the notion of a uniform base, that is, an open base B of the space X with the following property: for an arbitrary point $x \in X$ and an arbitrary neighbourhood Ox there exists in B only a finite number of elements containing the point x and having points in common with $X \setminus Ox$."

5025:

Stone, A. H. Cardinals of closed sets. Mathematika **6** (1959), 99–107.

Theorem: Let X be a metric space of cardinal number m and weight (this is the least cardinal number that a basis must have) $k \geq \aleph_0$. Then X has exactly 2^k closed subsets. Further, the number of closed sets of cardinal number n is (i) m^n if $n \leq k$, (ii) $2^k = k^k$ if $n = m$, and (iii) 0 if X is complete and $k < n < m$. Also, there exists at least one pair A, B of closed subsets of cardinal number m such that $A \cap B$ is at most one point. If $m > k$ then $m = \exp_k(\aleph_0)$. There exists a closed subset C of cardinal number $\exp_k(\aleph_0)$ which is homeomorphic to the product of countably many discrete spaces. These results are based on a study of the discrete subsets of metric spaces. These findings for metric spaces are especially remarkable because in the Čech compactification of the integers, a closed set is either finite or has the cardinal number 2^c.

R. Arens (Los Angeles, Calif.)

5026:

Dolcher, Mario. Topologie e strutture di convergenza. Ann. Scuola Norm. Sup. Pisa (3) **14** (1960), 63–92.

The author considers sets with a structure of convergence of sequences (of order type of positive integers) as objects of a convergence category $\mathscr{L}$. Maps of this category are those which preserve convergence. He subjects convergence of sequences to the usual axioms (a), (b) and (c) of J. L. Kelley, *General topology* [van Nostrand, New York, 1955; MR **16**, 1136; p. 74], but omits axiom (d) on iterated limits.

$\mathscr{L}$ is compared with the category $\mathscr{T}$ of topological spaces and continuous maps by considering two covariant functors $L: \mathscr{T} \to \mathscr{L}$ and $T: \mathscr{L} \to \mathscr{T}$. L is given by the usual definition of convergence. T assigns to a convergence structure λ the topology $T(\lambda)$ in which closed sets coincide with sets closed under convergence of sequences. Observe that a point of $\mathrm{Cl}\, A$ in $T(\lambda)$ need not be a limit of a sequence from A. A detailed study of composite functors of L and T follows, as well as a study of restrictions of these functors to various subcategories. E.g., it is shown that $LTL = L$, $TLT = T$, and that $LT|\mathscr{L}_2$ is the identity, where $\mathscr{L}_2$ denotes the subcategory of convergence structures with unique limits.

S. Mardešić (Zagreb)

5027:

Reiter, H. Über stetige, offene Abbildungen. Math. Ann. **140** (1960), 417–421.

Theorem: Every continuous, open mapping from a first-countable topological space onto a countable complete metric space has a continuous inverse (=cross section). *E. Michael* (Princeton, N.J.)

5028:

Nagata, J. On the countable sum of zero-dimensional metric spaces. Fund. Math. **48** (1959/60), 1–14.

Detailed proofs are given of the results contained in a

previous note [Proc. Japan Acad. **34** (1958), 146–149; MR **20** #6088]. In addition, strong countable-dimensional spaces (i.e., metrizable spaces R such that $R=\bigcup A_n$, A_n closed, $\dim A_n<\infty$) are considered, and a necessary and sufficient condition is given in terms of sequences of open coverings. It is proved that a space R with a σ-star-finite open base is a strong countable-dimensional space if and only if it is homeomorphic with a subspace of $N(\Omega)\times K_\omega$, where $N(\Omega)$ is a generalized Baire space (a product of countably many discrete spaces) and K_ω consists of all points $x=\{x_k\}$ of the cube I_ω such that $x_k=a_k$ for almost all k, $a=\{a_k\}\in I_\omega$ being fixed [for this result, in the case of a separable R, cf. also Yu. M. Smirnov, Izv. Akad. Nauk SSSR Ser. Mat. **23** (1959), 185–196; MR **21** #5178]. *M. Katětov* (Prague)

ALGEBRAIC TOPOLOGY

5029:

Dal Soglio, Letizia. Sulla nozione di grado e di coefficiente di allacciamento per mappe ponderate. Rend. Sem. Mat. Univ. Padova **28** (1958), 280–289.

Continuing the work of G. Darbo [same Rend. **28** (1958), 188–220; MR **21** #3838], the author defines notions of degree and linking coefficient. He proves suitable analogues of the basic theorems, such as the Brouwer fixed-point theorem. *D. W. Kahn* (New Haven, Conn.)

5030:

Kultze, Rolf. Dualität von Homologie- und Cohomologiegruppen in der Garbentheorie. Arch. Math. **10** (1959), 438–442.

This is an announcement of results (with no proofs). As the author points out, the basic notions of a co-presheaf (Cogarbendatum) $\mathfrak{G}$ and of the homology group $H_q(X,\mathfrak{G})$, defined in this paper, are the same as those introduced independently by E. Luft [Bonn. Math. Schr. no. 8 (1959); MR **21** #3841] (Luft calls $\mathfrak{G}$ an inverse group family). With every topological vector space sheaf $\mathfrak{G}$ (field of complex numbers) over a space X there is associated a topologically dual co-presheaf denoted by $\mathfrak{G}'\cdot\mathfrak{G}'$ is obtained from the vector spaces $\Gamma(U,\mathfrak{G})$ of sections of $\mathfrak{G}$ over open sets $U\subset X$ and homomorphisms $r_{UV}\colon\Gamma(U,\mathfrak{G})\to\Gamma(V,\mathfrak{G})$, $V\subset U$, by taking their topological duals. The author concentrates on $\mathfrak{F}$-sheaves, i.e., sheaves $\mathfrak{G}$ such that $\Gamma(U,\mathfrak{G})$ are complete metrizable locally convex vector spaces.

Several theorems expressing $H_q(X,\mathfrak{G}')$ in terms of $H^q(X,\mathfrak{G})$ are given. E.g.: Let X be a locally compact space with a countable basis at infinity. Let $\mathfrak{G}$ be an $\mathfrak{F}$-sheaf and let there be a cofinal system $\{U\}$ of coverings of X with finite-dimensional $H^q(U,\mathfrak{G})$, $q\geq 1$. Then $(H^q(X,\mathfrak{G}))'\cong H_q(X,\mathfrak{G}')$, $q\geq 0$ ($H^q(X,\mathfrak{G})$ is provided with the direct limit topology). If the coverings U of $\{U\}$ are Leray coverings, then a stronger form of this theorem can be obtained. In this case, the homology groups $H_q(X,\mathfrak{G}')$ are also expressed in terms of a resolution for $\mathfrak{G}$.

Next, some applications to complex manifolds are given, e.g.: If X is a compact complex manifold and $\mathfrak{G}$ a coherent analytic sheaf over X, then the algebraic dual of $H^q(X,\mathfrak{G})$ is canonically isomorphic with $H_q(X,\mathfrak{G}')$, for all $q\geq 0$. Finally, a Poincaré duality theorem for compact complex manifolds is stated, asserting the canonical isomorphism of the algebraic dual of $H_q(X,(\Omega^p(V))')$ with $H_{n-q}(X,(\Omega^{n-p}(V^*))')$; here V denotes a complex-analytic vector space bundle, V^* its dual bundle and $\Omega^p(V)$ denotes the sheaf of germs of locally holomorphic p-forms over X with coefficients from V. *S. Mardešić* (Zagreb)

5031a:

Yamazaki, Keijiro. Theory of sheaves. I. Sûgaku **7** (1955/56), 101–122. (Japanese)

5031b:

Yamazaki, Keijiro. Theory of sheaves. II. Sûgaku **8** (1956/57), 157–180. (Japanese)

This is a concise and lucid treatment of the theory of sheaves. The author has managed to select just those definitions which are necessary for readers unfamiliar with sheaves. The first part consists of an introduction and three chapters and the second contains two chapters. In the introduction he explains the historical background of sheaves and the circumstances of the latest developments of the theory. The first two chapters are devoted to some topics which brought about the concept of sheaf. The essential parts are concentrated in the remaining chapters. In general, the author has followed the viewpoint of H. Cartan in writing chapters III and IV. The last chapter gives an application of the theory to some important topics for manifolds. The following table of chapter headings and some selected topics treated in the chapters gives a rough idea of the content. Chapter I: Systems of local coefficients introduced by Steenrod. 1. Systems of local coefficients and bundles of coefficients. 2. Homology and cohomology. 3. An example (projective plane). Chapter II: Leray sheaves and gratings. 1. Definitions and fundamental concepts (substructures, homomorphisms and tensor products). 2. Examples (trivial and locally trivial Leray sheaves, gratings of differential forms, etc.). Chapter III: Definitions of sheaves and Čech cohomology. Chapter IV: Cohomology theory of sheaves. 1. Fine sheaves and examples. 2. The axiomatic construction of cohomology theory of sheaves. Chapter V: Fundamental theorems and their applications. 1. Spectral sequences. 2. Fundamental theorems for the cohomology group $H(\Gamma_\Phi(F))$, where $\Gamma_\Phi(F)$ is the group of sections of a sheaf F over a space X whose supports belong to a certain family Φ of subsets of X. 3. An application to the theory of manifolds (cohomology of HLC spaces, duality theorem of Poincaré, theorem of invariance of domain and theorem of de Rham). *A. Tominaga* (Kure)

5032:

Yamanoshita, Tsuneyo. On exact sequences of $A^*(Z_2,Z_2)$. Sûgaku **8** (1956/57), 33–37. (Japanese)

Let $A^*(Z_2,Z_2)=\sum_{i=0}^{\infty}A^i(Z_2,Z_2)$ be the algebra of "stable groups" [J.-P. Serre, Comment. Math. Helv. **27** (1953), 198–232; MR **15**, 643]. The author obtains an exact sequence of period 4:

$$\to A^*(Z_2,Z_2)\xrightarrow{\beta}A^*(Z_2,Z_2)\xrightarrow{\alpha}A^*(Z_2,Z_2)/\mathrm{Im}\,\mathrm{Sq}^1\xrightarrow{\delta}A^*(Z_2,Z_2)/\mathrm{Im}\,\mathrm{Sq}^1\xrightarrow{\gamma}A^*(Z_2,Z_2)\to,$$

with an extension of the first exactness. Cf. A. Negishi, J. Math. Soc. Japan **10** (1958), 71–78 [MR **20** #6093]. *H. Wada* (Sendai)

5033:

Takahashi, Tendai. On a chain-equivalent and product-preserving mapping from $S(X)$ to $\bar{Q}(X)$. Sûgaku **8** (1956/57), 37–39. (Japanese)

The author gives a direct proof of the fact that J. P. Serre's map θ_n of the n-cube I_n onto the n-simplex L_n [Ann. of Math. (2) **54** (1951), 425–505; MR **13**, 574] induces a chain equivalence $\theta_{\#n}$ of the singular complex $S_n(X)$ to the normalized cubical singular complex $\bar{Q}_n(X)$ of a topological space X and the cochain map $\theta_n^{\#}$ preserves products. In an earlier paper [Amer. J. Math. **75** (1953), 189–199; MR **14**, 670], S. Eilenberg and S. MacLane proved the chain equivalence by the theory of acyclic models. *H. Suzuki* (Sendai)

5034:

Nakaoka, Minoru. On cohomology of cyclic product spaces. Sûgaku **8** (1956/57), 72–83. (Japanese)

The contents are summarized in the author's paper, Proc. Japan Acad. **31** (1955), 665–669 [MR **19**, 972]; and the complete proofs are given in his next paper, J. Inst. Polytech. Osaka City Univ. Ser. A. **7** (1956), 51–102 [MR **19**, 972]. *H. Wada* (Sendai)

5035:

Mizuno, Katuhiko; Saito, Yoshihiro. Note on the relations on Steenrod algebra. Proc. Japan Acad. **35** (1959), 557–562.

The authors study some binomial coefficients modulo two or an odd prime and use the relations obtained and the Adem relations to prove theorems of the following form. (1) Let $M_2(k)=\sum_{i=0}^{k}\mathrm{Sq}^{k-i}\,\mathrm{Sq}^{i}$. Then $M_2(k)=0$ if $k\neq 3n$, and $\sum_{i=0}^{2n-1}\mathrm{Sq}^{i}\,\mathrm{Sq}^{3n-i}=0$. (2) Let c be the anti-automorphism of the Steenrod algebra. Then $\mathrm{Sq}^{a}+c(\mathrm{Sq}^{a})=\sum_{j=0}^{a-1}M_2(a-j)c(\mathrm{Sq}^{j})$. They also prove analogous theorems for odd primes.

F. P. Peterson (Cambridge, Mass.)

5036:

James, I. M. On the homotopy groups of spheres. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 222–224. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv+334 pp.

In the opening paragraph of this note, the author points out that there appear to be two kinds of general results about the higher homotopy groups of spheres. The first kind are relations between the homotopy groups of various spheres, such as the classical isomorphism $\pi_m(S^2)\approx\pi_m(S^3)$ for all $m\geq 3$, or various exact sequences involving homotopy groups of spheres. The second kind are theorems which give an indication, admittedly slight, of the general form taken by the homotopy groups of the q-sphere. This paper lists and discusses various theorems of the second kind. Most of these have been published elsewhere. Proofs are either omitted entirely or only briefly indicated.

W. S. Massey (New Haven, Conn.)

5037:

Piccinini, Renzo. On a product in n-dimensional varieties. Soc. Parana. Mat. Anuário (2) **1** (1958), 11–20. (Portuguese. English summary)

Using dual cell decompositions of a homology manifold (in the sense of Seifert and Threlfall), the author gives the definition of the intersection ring; method and results are similar to § 6 of Gysin, Comment. Math. Helv. **14** (1942), 61–122 [MR **3**, 317] except that the reference to the cup and cap products is not explicit. *G. Hirsch* (Brussels)

5038:

Smith, Paul A. Orbit spaces of abelian p-groups. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 1772–1775.

Let S be a cohomology n-sphere over Z_p (p prime), π_a an abelian group of order p^a and type $(p,\cdots,p)$ acting effectively on S; denote by S_f the π_a-free part of S (union of orbits of cardinality p^a). For $\pi_b\subset\pi_a$, denote by $F(\pi_b)$ the fixed-point set of π_b (cohomology sphere over Z_p) and write $n(\pi_b)$ for $\dim_p F(\pi_b)$. Cyclic subgroups π_1 not in π_b act effectively on $F(\pi_b)$. Borel [Comment. Math. Helv. **29** (1955), 27–39; MR **16**, 611] proved that, if $a=1$, the Poincaré polynomial of the cohomology over Z_p of the orbit space S_f/π_1 is $t^{n(\pi_1)+1}+t^{n(\pi_1)+2}+\cdots+t^{n}$.

The author conjectures that if $a\geq 2$, for any subgroup π_{a-1} of π_a, $(1-t)$ times the Poincaré polynomial over Z_p of S'/π_a is equal to the sum (for $\pi_1\not\subset\pi_{a-1}$) of Poincaré polynomials (over Z_p) of F_f/π_{a-1} minus t times the Poincaré polynomial of S'/π_{a-1}, where F_f denotes the π_{a-1}-free part of $F(\pi_1)$ and S' the π_{a-1}-free part of S. The sketch of the proof is given in the case $a=2$, where the $p+1$ subgroups π_{a-1} are cyclic and where π_a/π_1 is also a cyclic group β of order p; it uses a free $Z_p(\beta)$-module which is a cochain complex (β being a cyclic group of order p, and $Z_p(\beta)$ the group ring). *G. Hirsch* (Brussels)

5039:

Noguchi, Hiroshi. The thickening of combinatorial n-manifolds in $(n+1)$-space. Proc. Japan Acad. **36** (1960), 70–71.

This paper deals with the embedding of a compact orientable combinatorial n-manifold M in an orientable $(n+1)$-manifold W without boundary by a piecewise linear map. Suppose we assume the generalized Schoenflies conjecture in dimension $\leq n$; i.e., we assume that a combinatorial p-sphere in Euclidean space R^{p+1} bounds a $(p+1)$-cell if $p<n$. Then there is a piecewise linear homeomorphism $\theta: M\times J\to W$ such that $\theta(x,0)=x$, where J denotes the interval $[-1,1]$. Moreover, the image $\theta(M\times J)$ lies in a certain pre-assigned neighbourhood of M. Several related theorems are also stated.

V. Gugenheim (Baltimore, Md.)

5040:

Gardner, Martin. The celebrated four-color map problem of topology. Sci. Amer. **203** (1960), no. 3, 218–226.

Popular exposition.

DIFFERENTIAL TOPOLOGY

See also 5012, 5030.

5041:

Bauer, Friedrich-Wilhelm. Tangentialstrukturen. . Ann. Inst. Fourier. Grenoble **9** (1959), 111–146.

This paper is very abstract. The author defines a

"tangential structure P with norm-domain A and atom domain N" to be a partially ordered set P with a null element, a subset N of P, a lattice A with null element, and a mapping $a \to |a|$ of P into A such that the following axioms hold. (1) If $a \leq b$ in P, then $|a| \leq |b|$. (2) If $a \leq b$ in P and $|a| = |b|$, then $a = b$. (3) If $z \in N$, $a \neq 0$ in P and $a \leq z$, then $a = z$. (4a) If $a \neq 0$ in P, then there exists $z \leq a$ in N. (4b) For any $z \in N$ there exists an $a \in P - N - \{0\}$ such that $z < a$. (5a) If $a \leq b_1, b_2$ and $|b_1| = |b_2|$, then $b_1 = b_2$. (5b) If $a \in P - N - \{0\}$, and $X \geq |a|$, then there exists a unique $b \in P - N - \{0\}$ such that $a \leq b$ and $|b| = X$.

A tangential structure in the sense of this definition may be thought of as a generalization of a homology structure as defined in two earlier papers of the author [Math. Ann. **135** (1958), 93–114; **136** (1958), 348–364; MR **20** #4262; **21** #3842]. The author asserts that he is not greatly interested in a purely algebraic lattice-theoretic treatment of the theory of tangential structures. The axioms were concocted with certain definite geometric examples in mind. The most important of these examples are described in the paper; they are much too involved to give here. One of the examples involves manifolds differentiably imbedded in Euclidean space, hence the name "tangential structure".

The author promises applications of the theory in a subsequent paper. *W. S. Massey* (New Haven, Conn.)

5042:

Bauer, Friedrich-Wilhelm. Algebraische Übertragungen. Math. Ann. **139**, 31–43 (1959).

This paper is a complement to a previous paper by the author on "tangential structures" [see the preceding review]. Its purpose is described by the author in the introduction as follows:

"Die Theorie der Tangentialstrukturen hat in der Frage ihren Ursprung, ob man gewisse differentialgeometrische Begriffsbildungen mit Erfolg auf andere, algebraisch definierte Strukturen übertragen kann, in denen von Differenzierbarkeit keine Rede ist. Ein solcher wichtiger Spezialfall einer Tangentialstruktur ist z.B. der einer Homologiestruktur, der von der einer differenzierbaren denkbar weit entfernt ist. Von besonderer Bedeutung in der modernen Differentialgeometrie ist nun der Begriff der Übertragung (connection), der hier für eine beliebige Tangentialstruktur behandelt werden soll. Zum Unterschied von den differentialgeometrischen Übertragungen wollen wir hier von einer 'algebraischen Übertragung' sprechen, wenn wir eine Übertragung in einer allgemeinen Tangentialstruktur vor Augen haben. Die Übertragungen in einer Tangentialstruktur, die algebraischen Übertragungen, stehen mehr am Rande der allgemeinen Theorie der Tangentialstrukturen, auf deren wichtigste Ergebnisse wir weiter unten noch hinweisen wollen." *W. S. Massey* (New Haven, Conn.)

5043:

Weier, Josef. Über Felder in offenen Riemannschen Mannigfaltigkeiten. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1959**, 13–22 (1960).

It is shown, first, that a continuous differentiable field ϕ of tangent vectors, without zero points, exists on an arbitrary open connected Riemannian manifold. It is then shown that, given ϕ, there exists a second such field orthogonal to ϕ, if and only if a certain cycle, whose coefficients are rotation numbers, is homologous to zero. Reference is made to some, but not all, of the related work by Whitney and others. *S. S. Cairns* (Urbana, Ill.)

5044:

Weier, J. Die Charakteristik verallgemeinerter Vektorfelder. Collect. Math. **11** (1959), 61–68.

Given two closed orientable manifolds P^{n+1} (triangulated) and Q^n (differentiable) and given a map $f: P^{n+1} \to Q^n$, the author considers maps φ of P^{n+1} into the tangent bundle $T(Q^n)$ such that φ covers f. He states that by geometric arguments of the kind used in his paper in Rend. Circ. Mat. Palermo (2) **6** (1957), 343–348 [MR **20** #4271] one can prove existence of maps φ for which $\varphi^{-1}(0)$ is a 1-sphere x. A degree associated with x and φ defines a 1-homology class ξ on P^{n+1}. ξ is shown independent of the choice of φ and invariant under homotopy of f. *S. Mardešić* (Zagreb)

5045:

Takizawa, Seizi. On characteristic classes of sphere bundles. Sûgaku **8** (1956/57), 229–245. (Japanese)

The subject is an exposition on the subject of the title. In the first place, the Stiefel-Whitney, Pontrjagin, Euler-Poincaré, Chern and their dual classes of a sphere bundle over a complex are defined as counter images of the cohomology classes of the Grassmann manifolds considered as base spaces of the Stiefel manifolds. Next, the characteristic classes of some dimensions are considered as the obstruction classes of the cross-sections. In the next place, for the differentiable manifold M of class C^∞, connection and curvature forms are defined; and the characteristic classes of a sphere bundle over M are defined by means of the curvature form. Also some results on the admissibility of almost complex structure and on the imbeddability of some differentiable manifolds in Euclidean spaces are reported. *H. Wada* (Sendai)

5046:

Tamura, Itiro. Characteristic classes of M-spaces. I. J. Math. Soc. Japan **11** (1959), 312–342.

The author generalizes the notion of manifold with singularities as follows. Let $\tilde{M}$ be a connected, C^∞-manifold. Let $W = {}^1W \cup \cdots \cup {}^lW$, where ${}^iW$ is a (not necessarily connected) submanifold of $\tilde{M}$. Let G be a finite group of C^∞-automorphisms of W sending each ${}^iW$ into itself. Let $M = (\tilde{M} - W) \cup (W/G)$. The collection is called a C^∞-M-space. A C^∞-D-bundle over a C^∞-M-space is a fibre bundle B over $\tilde{M}$ with fibre F, a collection of fibre bundles ${}^iB$ over ${}^iW$ with fibres ${}^iF \subset F$, and C^∞-injections of ${}^iB \to B|{}^iW$. With a number of additional assumptions, the author does obstruction theory for such D-bundles and defines the analogs of Stiefel-Whitney classes, Pontrjagin classes and Chern classes (called SW-classes, P-classes, and C-classes, respectively). If M is compact and $W = \tilde{M}$ and under further assumptions, he shows that the "Euler characteristic" is a homotopy-type invariant. *F. P. Peterson* (Oxford)

5047:

Tamura, Itiro. Characteristic classes of 2-fold symmetric products of spheres. J. Math. Soc. Japan **12** (1960), 8–15.

$S^n * S^n$, the symmetric product of an n-sphere with itself, is a $2n$-dimensional C^∞-M-space in the sense of the preceding review. In this paper, the author calculates some characteristic classes for $S^n * S^n$, in the sense of the preceding paper. He obtains the following results: (1) $C_1(S^2 * S^2) = 3\mu_2$, $C_2(S^2 * S^2) = 3\mu_4$; (2) $P_1(S^4 * S^4) = 2\mu_4$, $P_2(S^4 * S^4) = \mu_8$; (3) $P_2(S^8 * S^8) = 6\mu_8$, $P_4(S^8 * S^8) = \mu_{16}$. *F. P. Peterson* (Oxford)

5048:

Tamura, Itiro. A diffeomorphy invariant of quotient manifolds. J. Math. Soc. Japan **11** (1959), 343–353.

Let $\tilde{M}$ be a compact, orientable, $(2n+1)$-dimensional, C^∞-manifold and G a finite group of fixed-point-free C^∞-transformations on $\tilde{M}$. Assume $\tilde{M}$ is the boundary of a compact, connected, oriented, $(2n+2)$-dimensional manifold, N. Using N and the mapping cylinder of $p: \tilde{M} \to M = \tilde{M}/G$, the author defines a $(2n+2)$-dimensional M-space [#5046 above] and defines $\Pi(M)$ in terms of the $(n+1)/2$-P-class of the M-space. $\Pi(M)$ is an integer mod m, where m is the order of G. Theorem: If $\tilde{M}$ is the universal covering manifold of M and some further conditions hold, then $\Pi(M)$ is independent of the choices involved and is a diffeomorphy invariant of M. In particular, the conditions are satisfied when $\tilde{M} = S^{2n+1}$, and the author uses his invariant to study lens spaces. For example, he proves the following theorem: $\Pi(L^3(m; r)) \equiv (r + r^{-1}) \bmod m$. *F. P. Peterson* (Oxford)

5049:

Look, K. H. Explicit formula of the infinitesimal connection for a continuous representation of GL(n, R). Sci. Record (N.S.) **3** (1959), 295–300.

Es sei V_n eine Riemannsche Mannigfaltigkeit. Verf. gibt explizite Formeln an für infinitesimale Zusammenhänge in den Vektorraum-Bündeln über V_n, die zum Tangentialbündel vermöge Darstellungen GL(n, R)$\to$GL(N, C) assoziiert sind. Es genügt, diese expliziten Formeln für die Potenzen der Determinantendarstellung, für ganzrationale Darstellungen und für gewisse nicht vollständig reduzible Darstellungen herzuleiten. *F. Hirzebruch* (Bonn)

5050:

Kashiwabara, Shôbin. The structure of a Riemannian manifold admitting a parallel field of tangent vector subspaces. Tôhoku Math. J. (2) **12** (1960), 102–119.

This paper continues the study of Riemannian manifolds admitting parallel distributions and expresses in a different and in some cases more general form some of the theorems given by the reviewer on the fibering of such manifolds [Proc. London Math. Soc. (3) **3** (1953), 1–19; MR **15**, 159]. It pays particular attention to the relation between the topology of the manifold and that of the maximal integral sub-manifolds of the distribution. *A. G. Walker* (Liverpool)

5051:

Cattaneo-Gasparini, Ida. Sulle connessioni proiettive. Ann. Mat. Pura Appl. (4) **50** (1960), 467–473.

The present paper associates with every differentiable variety V_n, of class C^∞, a fibre space $P(V_n)$ having V_n as basis and the projective group in $n+1$ homogeneous variables as structural group. Then the projective and affine tensors are defined, and the projective connexions on V_n are introduced as infinitesimal connexions on $P(V_n)$, which leads to a natural explanation for the duality between affine and coaffine connexions. Finally, it is shown that every regular projective connexion defines canonically an affine connexion, the two connexions having the same geodesics. Also the geodesics for a general projective connexion are obtained, in a way which associates with the connexion a linear connexion, an affine tensor and n Pfaffian forms. *B. Segre* (Rome)

5052:

Smith, J. Wolfgang. Fundamental groups on a Lorentz manifold. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 111–114.

The author describes a refinement of the fundamental group for Lorentz manifolds (metric with one time-like and $n-1$ space-like directions). A q-loop at x is a piecewise differentiable path f such that (1) $f^{-1}(x) = \{0, 1\}$, (2) all tangent vectors of f are time-like, and (3) there are exactly q corners (x not among them) where the two tangent vectors point in opposite time directions. A sting is a path from x to x of the form $g \cdot g^{-1}$. The group τ_q is now defined just like π_1 but staying completely in the set of paths that are finite products of q-loops and stings. There are natural maps into π_1, and maps $h_q: \tau_q \to \tau_{q+2}$. The two limit groups associated with $\tau_0 \to \tau_2 \to \cdots$ and $\tau_1 \to \tau_3 \to \cdots$ are isomorphic with the normal subgroups of π_1 generated by the elements whose holonomy map preserves or reverses, respectively, the time direction at x. Among the results is the following. If the manifold is flat and complete, then h_0 is injective and the h_q, $q > 0$, are bijective, thus connecting the new groups with global curvature effects. *H. Samelson* (Stanford, Calif.)

5053:

Adams, J. Frank. Exposé sur les travaux de C. T. C. Wall sur l'algèbre de cobordisme Ω. Bull. Soc. Math. France **87** (1959), 281–284.

Sketch of work of Wall in Bull. Amer. Math. Soc. **65** (1959), 329–331 [MR **21** #6586]. *N. Stein* (New York)

5054:

Kervaire, Michel A. Sur le fibré normal à une variété plongée dans l'espace euclidien. Bull. Soc. Math. France **87** (1959), 397–401.

This note is based on an address given by the author at a topology colloquium held at Lille in the summer of 1959. In it the author summarizes some of the facts known about the normal bundle to a manifold imbedded differentiably in Euclidean space, and proves some new theorems on this subject. Among these, the following may be mentioned. (a) The normal bundle to a d-dimensional compact orientable differentiable manifold imbedded in $(n+d)$-dimensional Euclidean space is independent of the imbedding for $n \geq d+1$. (This was previously known for $n \geq d+2$.) (b) Every imbedding of a d-dimensional π-manifold in $2d$-dimensional Euclidean space has a trivial

normal bundle (by definition, a π-manifold has trivial normal bundle when imbedded in a sufficiently high dimensional Euclidean space).

W. S. Massey (New Haven, Conn.)

5055:

Atiyah, Michael F.; Hirzebruch, Friedrich. Quelques théorèmes de non-plongement pour les variétés différentiables. Bull. Soc. Math. France **87** (1959), 383–396.

If X is a space, $\mathrm{ch}(X) \subset H^*(X; Q)$ denotes the subring of the rational (even-dimensional) cohomology ring which is generated by the characters of complex vector-bundles over X. In their paper, Bull. Amer. Math. Soc. **65** (1959), 276–281 [MR **22** #989], the authors derived certain integrality conditions concerning the position of $\mathrm{ch}(X)$ in $H^*(X; Q)$ when X is a manifold (always connected, orientable and compact) whose third Whitney-Stiefel class w_3 vanishes. In the present paper they find new integrality conditions when the assumption $w_3 = 0$ is replaced by the hypothesis that X is a $2n$-dimensional submanifold of S_{2n+2k}. Using both these criteria they then can derive non-imbeddability theorems. For instance, they show that $P_n(C)$—the complex projective n-space—cannot be imbedded in $S_{4n-2\alpha(n)}$, where $\alpha(n)$ denotes the number of powers of 2 needed to express n in the binary form. This is a considerable improvement on previous bounds.

The integrality conditions on $\mathrm{ch}(X)$ are conveniently recorded in terms of the "Hilbert polynomials" $H(t)$ associated to a pair $d \in H^2(X; Z)$ and $z \in \mathrm{ch}(X)$. By definition, $H(t) = \hat{A}(\mathcal{T}_X) \cdot e^{d/2} \cdot z^{(t)}[X]$ where: $z \to z^{(t)}$ means multiplication of the $(2n)$th component by t^n; $\hat{A}$ is the unique multiplicative function on real bundles which on an $O(2)$-bundle, ξ, has the value $\sinh(\frac{1}{2}x)/\frac{1}{2}x$, $x = c_1(\xi)$; $\mathcal{T}_X$ is the tangent bundle of X; and $[X]$ stands for the orientation class of X. In this terminology, the earlier integrality theorem states that if $d \equiv w_2(X) \bmod 2$ then the polynomial $H(t)$ has integral values for $z \in \mathrm{ch}(X)$ and $t \in Z$. The new integrality theorem states that when $X \subset S_{2n+2k}$, $\dim X = 2n$ $(k > 0)$, then for any $d \in H^2(X; Z)$ and $z \in \mathrm{ch}(X)$ the expression $2^{n+k-1} H(\frac{1}{2})$ is an integer.

The proof of both these theorems is based on the same principle. In the present case the argument takes this form: Let W be a normal tube about $X \subset S_{2n+2k}$; thus W can be considered as an $SO(2k)$-bundle η over X. When η is lifted to ∂W its structure group can be reduced to $SO(2k-1)$. Suppose now that ρ_1 and ρ_2 are two unitary representations of $SO(2k)$ which are isomorphic on $SO(2k-1)$. It follows then that the complex bundles associated to η by ρ_1 and ρ_2, say $\rho_1(\eta)$ and $\rho_2(\eta)$, admit an isomorphism ϕ when lifted to ∂W. But such an isomorphism determines a difference bundle $\{\phi\}$ on $W/\partial W$, and hence, by extending with the trivial bundle, on S_{2n+2k}. Now one computes $\mathrm{ch}\{\phi\}$ in the universal case and finds that it takes the form $i_* \cdot u_\phi$, where $u_\phi \in H^*(X; Q)$ and i_* is the Gysin homomorphism of the injection $X \to S_{2n+2k}$. More generally, if ξ is any U-bundle on X, the isomorphism $\phi \otimes 1$ of $\rho_1(\eta) \otimes \xi$ and $\rho_2(\eta) \otimes \xi$, lifted to ∂W, gives rise to a difference bundle $\{\phi; \xi\}$ on S_{2n+2k} with $\mathrm{ch}\{\phi; \xi\} = i_*\{u_\phi \cdot \mathrm{ch}(\xi)\}$. However, it is known that the character of any bundle on the sphere takes an integral value on S_{2n+2k}. Thus $u_\phi \cdot \mathrm{ch}(\xi) \cdot [X]$ must always be an integer. In the present instance the representations ρ_1 and ρ_2 are so chosen that in the representative ring $RU\,SO(2k)$ (the free module generated by the irreducible complex $SO(2k)$-modules) $\rho_1 - \rho_2$ is represented by $\mu_+ = \sum_{i=1}^{k-1} (-1)^i \lambda^i + (-1)^k \lambda_+^k$, where the λ^i are the exterior powers of complexification of the standard representation module of $SO(2k)$, and λ_+^k is one of the two irreducible parts into which the module λ^k splits. Using this description of $\rho_1 - \rho_2$, the authors compute u_ϕ in the universal case, and translate the integrality condition just stated into the earlier formulation. (In the integrality theorem which we stated first, one takes $X \subset S_{2n+2k}$, k large, reduces the structure group of η to $\mathrm{Spin}(2k) \times_{Z_2} S^1$—which is possible precisely because $w_2 = 0$—and then uses the two spin representations Δ^+ and Δ^- for ρ_1 and ρ_2.

Under certain additional conditions on the dimensions of X and k the divisibility theorem already stated can be improved by a factor of 2. This arises from the fact that sometimes the "representation" μ_+ comes from a real module or a symplectic module, thereby enabling the authors to keep track of the subring $\mathrm{ch}\, O(X)$ [$\mathrm{ch}\, SP(X)$] which corresponds to the characters of O-bundles [Sp-bundles].

R. Bott (Cambridge, Mass.)

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